On some reducible representations of the quantized coordinate algebras

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Subjects

Main objects: Representations of quantized coordinate algebras for finite dimensional simple Lie algebras over $\mathbb{C}$.

For any sequence $\mathbf{i}$ of the indices of simple roots, we can construct a module $V_{\mathbf{i}}$ over the quantized coordinate algebra.

Theorem (Soibelman ('90))

The module $V_{\mathbf{i}}$ is irreducible if and only if $\mathbf{i}$ is a reduced word of an element of the Weyl group.

(Naive) problem: Describe the structure of $V_{\mathbf{i}}$ in the case that $\mathbf{i}$ does not correspond to a reduced word.

How do we "describe" $V_{\mathbf{i}}$?

Key: Kuniba-Okado-Yamada's result ('13);
There exists a "natural" (but somewhat mysterious) isomorphism between

- the module $V_{\mathbf{i}_0}$ corresponding to the longest element w_0, and
- (the positive half of) the quantized enveloping algebra U^+.

Through this result, we can say that

"the structure of $V_{\mathbf{i}_0}$ is described by the algebra structure of U^+".

$\rightsquigarrow$ How about the other $\mathbf{i}$'s?
<u>Our target</u>: The modules $V_{\tilde{\mathbf{i}}}$ for $\tilde{\mathbf{i}} = (i_l, \dots, i_1, i_1, \dots, i_l)$ with $\overline{(i_1, \dots, i_l)}$ is a reduced word of an element of the Weyl group.
(In particular, they include the case $V_{(i,i)}$.)

Quantized enveloping algebras

Basic notation: Let

- $\mathfrak{g}$ a symmetrizable Kac-Moody Lie algebra($\supset$ finite dimensional simple Lie algebra) over $\mathbb{C}$,
- P the weight lattice of $\mathfrak{g}$ and $\{\alpha_i^{(\vee)}\}_{i\in I}$ the simple (co)roots of $\mathfrak{g}$,
- $U_q(:= U_q(\mathfrak{g}))$ the quantized enveloping algebra over $\mathbb{Q}(q)$;
  Chevalley generator: E_i, F_i $(i \in I)$, K_h $(h \in P^*)$,
  Some relations: $K_h E_i = q^{\langle \alpha_i, h\rangle} E_i K_h$, q-Serre relations, ...
  (This is a q-analogue of $U(\mathfrak{g})$),
- $U_q^{+(\text{resp. } -)}$ the subalgebra of U_q generated by E_i's (resp. F_i's).

Hopf algebra structure of U_q $(K_i := K_{\frac{(\alpha_i,\alpha_i)}{2}\alpha_i^\vee})$

$$\Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i,\ \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i,\ \Delta(K_h) = K_h \otimes K_h,$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \varepsilon(K_h) = 1, \exists \text{antipode}.$$

Quantized coordinate algebras

Definition

The quantized coordinate algebra $A_q[\mathfrak{g}]$ is a subalgebra of U_q^* generated (in fact, spanned) by the matrix coefficients

$$c^{\lambda}_{f,v}(:= c^{V(\lambda)}_{f,v}) := (X \mapsto \langle f, X.v\rangle),$$

here

- $\lambda \in P_+ := \{\mu \in P \mid \langle \mu, \alpha_i^\vee \rangle \geq 0 \text{ for all } i \in I\}$,
- $V(\lambda)$ the integrable highest weight U_q-module with highest weight λ,
- $v \in V(\lambda)$, $f \in V(\lambda)^*_{\mathrm{gr}}$(the graded dual of $V(\lambda)$).

In $A_q[\mathfrak{g}]$, we have

$$c^{\lambda}_{f,v} c^{\lambda'}_{f',v'} = c^{V(\lambda)\otimes V(\lambda')}_{f\otimes f', v\otimes v'}.$$

An important example

The quantized coordinate algebra $A_q[\mathfrak{sl}_2]$ of $\mathfrak{sl}_2$ is isomorphic to the $\mathbb{Q}(q)$-algebra generated by c_{ij} $(i, j \in \{1, 2\})$ with the following relations; (If "$q = 1$", then the first six relations are the same.)

$$\begin{gathered} c_{i1}c_{i2} = qc_{i2}c_{i1} \ (i = 1, 2), \quad c_{1j}c_{2j} = qc_{2j}c_{1j} \ (j = 1, 2), \\ [c_{12}, c_{21}] = 0, \qquad [c_{11}, c_{22}] = (q - q^{-1})c_{12}c_{21}, \\ c_{11}c_{22} - qc_{12}c_{21} = 1. \end{gathered}$$

Indeed the isomorphism of algebras is given by

$$c_{ij} \mapsto c^{V(\varpi)}_{f_i, v_j},$$

where $V(\varpi)$ is an irreducible 2-dimensional representation of $U_q(\mathfrak{sl}_2)$ with a highest weight vector v_1 and $v_2 = F.v_1$, and $\{f_1, f_2\}$ is its dual basis.

The module $V_\star$ of $A_q[\mathfrak{sl}_2]$

We have an $A_q[\mathfrak{sl}_2]$-module $V_\star := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) |m\rangle$ given by

$$\begin{aligned}
c_{11}. : |m\rangle &\longmapsto \begin{cases} 0 & \text{if } m = 0, \\ |m-1\rangle & \text{if } m \in \mathbb{Z}_{>0}, \end{cases} \\
c_{12}. : |m\rangle &\longmapsto -q^{m+1} |m\rangle \quad \text{for } m \in \mathbb{Z}_{\geq 0}, \\
c_{21}. : |m\rangle &\longmapsto q^{m} |m\rangle \quad \text{for } m \in \mathbb{Z}_{\geq 0}, \\
c_{22}. : |m\rangle &\longmapsto (1 - q^{2(m+1)}) |m+1\rangle \quad \text{for } m \in \mathbb{Z}_{\geq 0}.
\end{aligned}$$

By its construction, it is easy to see that this is an irreducible $A_q[\mathfrak{sl}_2]$-module.

The module $V_{\mathbf{i}}$ of $A_q[\mathfrak{g}]$

For $i \in I$, we denote by $U_{q_i,i}$ the Hopf subalgebra of U_q generated by $\{E_i, F_i, K_i^{\pm 1}\}$.

The Hopf algebra $U_{q_i,i}$ is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$ ($q_i := q^{\frac{(\alpha_i,\alpha_i)}{2}}$).

$\rightsquigarrow$ We can regard $\mathbb{Q}(q) \otimes_{\mathbb{Q}(q_i)} A_{q_i}[\mathfrak{sl}_2]$ as a subalgebra of $U_{q_i,i}^*$ and denote this subalgebra by A_i.

The irreducible module $V_\bigstar$ corresponding to A_i will be denoted by $V_i := \oplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) \left| m \right\rangle_i$.

For $\mathbf{i} = (i_1, \ldots, i_l) \in I^l$, set $m_{\mathbf{i}} : U_{q_{i_1},i_1} \otimes \cdots \otimes U_{q_{i_l},i_l} \xrightarrow{\text{multiplication}} U_q$.

$\rightsquigarrow$ $\exists$ the algebra homomorphism $m_{\mathbf{i}}^* : A_q[\mathfrak{g}] \to A_{i_1} \otimes \cdots \otimes A_{i_l}$.

Via $m_{\mathbf{i}}^*$, the $\mathbb{Q}(q)$-vector space $V_{\mathbf{i}} := V_{i_1} \otimes \cdots \otimes V_{i_l}$ is regarded as an $A_q[\mathfrak{g}]$-module.

The case: $\mathbf{i}$ is a reduced word

Let W be the Weyl group of $\mathfrak{g}$.
For $w \in W$, denote by $I(w)$ the set of the reduced words of w.
(ex. $\mathfrak{g} = \mathfrak{sl}_3$, w_0 the longest element, $I(w_0) = \{(1,2,1),(2,1,2)\}$.)

Theorem (Soibelman, Narayanan, Tanisaki, (O-))

The $A_q[\mathfrak{g}]$-module $V_{\mathbf{i}}$ is irreducible if and only if $\mathbf{i} \in I(w)$ for some $w \in W$.
Moreover, for $\mathbf{i}_1, \mathbf{i}_2 \in I(w)$, there is an isomorphism of $A_q[\mathfrak{g}]$-modules $V_{\mathbf{i}_1} \to V_{\mathbf{i}_2}$ given by

$$|(0)\rangle_{\mathbf{i}_1} \mapsto |(0)\rangle_{\mathbf{i}_2}.$$

Hence we denote the module $V_{\mathbf{i}}$ $(\mathbf{i} \in I(w))$ by V_w. (i. e. The modules $\{V_{\mathbf{i}}\}_{\mathbf{i} \in I(w)}$ are identified via the above isomorphism.)

KOY's isomorphism: preliminaries

Definition (The quantum nilpotent subalgebras $U_q(w)^{\pm}$)

For $w \in W$ and $\mathbf{i} = (i_1, \ldots, i_l) \in I(w)$, set

$$E_{\mathbf{i}}^{\mathbf{c}} := E_{i_l}^{(c_l)} T'_{i_l,1}(E_{i_{l-1}}^{(c_{l-1})}) \cdots T'_{i_l,1} \cdots T'_{i_2,1}(E_{i_1}^{(c_1)}), \text{ and}$$
$$F_{\mathbf{i}}^{\mathbf{c}} := F_{i_l}^{(c_l)} T''_{i_l,1}(F_{i_{l-1}}^{(c_{l-1})}) \cdots T''_{i_l,1} \cdots T''_{i_2,1}(F_{i_1}^{(c_1)}),$$

where $\mathbf{c} = (c_1, \ldots, c_l) \in \mathbb{Z}_{\geq 0}^l$, $X_i^{(c)}$ denotes the divided power and $T'_{i_l,1}, T''_{i,1}$ are q-analogues of the actions of the braid group.
The set $\{E_{\mathbf{i}}^{\mathbf{c}}(\text{resp. } F_{\mathbf{i}}^{\mathbf{c}})\}_{\mathbf{c}}$ is a linearly independent set of $U_q^{+(\text{resp. } -)}$.
Let us denote by $U_q^{+(\text{resp. } -)}(w)$ the $\mathbb{Q}(q)$-vector subspace of $U_q^{+(\text{resp. } -)}$ spanned by $\{E_{\mathbf{i}}^{\mathbf{c}}(\text{resp. } F_{\mathbf{i}}^{\mathbf{c}})\}_{\mathbf{c}}$. Here $U_q^{\pm}(e) = \mathbb{Q}(q)(\subset U_q^{\pm})$.
In fact these subspaces do not depend on the choice of $\mathbf{i} \in I(w)$.

ex. $\mathfrak{g} = \mathfrak{sl}_3$, $X_{(1,2,1)}^{(1,1,2)} = X_1^{(2)}(X_2X_1 - qX_1X_2)X_2$ $(X = E, F)$.

KOY's isomorphism

Let $U_q^{\pm}(w)^{\perp}$ be the orthogonal complements of $U_q^{\pm}(w)$ with respect to the Lusztig type bilinear forms on $U_q^{\pm}$. Then they are left ideals of $U_q^{\pm}$.
Denote the quotient maps $U_q^{\pm} \to U_q^{\pm}/U_q^{\pm}(w)^{\perp}$ by $X \mapsto [X]_w$. Note that $\{[E_{\mathbf{i}}^{\mathbf{c}}]_w(\text{resp. } [F_{\mathbf{i}}^{\mathbf{c}}]_w)\}_{\mathbf{c}}$ is a basis of $U_q^{\pm}/U_q^{\pm}(w)^{\perp}$ respectively.

Theorem (Kuniba-Okado-Yamada, Saito, Tanisaki, (O-))

Let $\mathbf{i} \in I(w)$. Define the $\mathbb{Q}(q)$-linear isomorphism

$$\Phi_{\mathrm{KOY}}^{+,w} : V_w \to U_q^{+}/U_q^{+}(w)^{\perp} \text{ by } |\mathbf{c}\rangle_{\mathbf{i}} \mapsto [E_{\mathbf{i}}^{\mathbf{c}}]_w,$$

where $|\mathbf{c}\rangle_{\mathbf{i}} := |c_1\rangle_{i_1} \otimes \cdots \otimes |c_l\rangle_{i_l}$. Then $\Phi_{\mathrm{KOY}}^{+,w}$ is well-defined. (i. e. does not depend on the choice of $\mathbf{i} \in I(w)$.)
Moreover, when we identify V_w with $U_q^{+}/U_q^{+}(w)^{\perp}$ via $\Phi_{\mathrm{KOY}}^{+,w}$, the multiplication operator $E_i\cdot$ is explicitly written in terms of the matrix coefficients.

Towards the non-reduced word case

Technical modification:
$\check{U}_q^0$:=the group algebra of P over $\mathbb{Q}(q)$ $(=: \bigoplus_{\lambda \in P} \mathbb{Q}(q) K_\lambda)$.
Set $\check{U}_q^{\gtreqless 0} := U_q^{\pm} \check{U}_q^0$. (i. e. $K_\lambda E_i = q^{(\alpha_i, \lambda)} E_i K_\lambda$ etc.)
Note that $U_q^{\pm} / U_q^{\pm}(w)^{\perp}$ can be regarded as $\check{U}_q^{\gtreqless 0}$-modules by $K_\lambda . [1]_w = [1]_w$ for all $\lambda \in P$.

Key Tool

There exists an embedding $\Omega : A_q[\mathfrak{g}] \to \check{U}_q^{\leq 0} \otimes \check{U}_q^{\geq 0}$ of $\mathbb{Q}(q)$-algebras using the Drinfeld pairing (cf. Joseph's textbook).

An example

Let $\mathfrak{g} = \mathfrak{sl}_2$. The weight lattice of $\mathfrak{sl}_2$ is written as $P = \mathbb{Z}\varpi$ (ϖ is the fundamental weight). Then

$$\begin{aligned}
\Omega(c_{11}) &= K_{-\varpi} \otimes K_{\varpi}, \\
\Omega(c_{12}) &= (1-q^2)K_{-\varpi} \otimes EK_{\varpi}, \\
\Omega(c_{21}) &= (1-q^2)FK_{-\varpi} \otimes K_{\varpi}, \\
\Omega(c_{22}) &= (1-q^2)^2 FK_{-\varpi} \otimes EK_{\varpi} + K_{\varpi} \otimes K_{-\varpi}.
\end{aligned}$$

(Note that $K_{\varpi}E = qEK_{\varpi}, K_{\varpi}F = q^{-1}FK_{\varpi}$)
$\rightsquigarrow \mathcal{S} := \{c_{11}^n\}_{n\geq 0}$ is an Ore set in $A_q[\mathfrak{sl}_2]$ and $A_q[\mathfrak{sl}_2][\mathcal{S}^{-1}]$ is isomorphic to $\bigoplus_{n\in\mathbb{Z}} U_q^- K_{-n\varpi} \otimes U_q^+ K_{n\varpi}$.

The case: i is of the form $\tilde{\mathrm{i}}$

Via Ω, the $\check{U}_q^{\leq 0} \otimes \check{U}_q^{\geq 0}$-module $U_q^-/U_q^-(w)^\perp \otimes U_q^+/U_q^+(w)^\perp$ can be regarded as an $A_q[\mathfrak{g}]$-module.

Theorem (O-)

Let $w \in I(w)$. Define $\tilde{V}_w := (V_{w^{-1}} \otimes V_w)^{\mathrm{tw}}$ ($\leftarrow$ slightly twisted). Then $\tilde{V}_w$ is isomorphic to $U_q^-/U_q^-(w)^\perp \otimes U_q^+/U_q^+(w)^\perp$ as an $A_q[\mathfrak{g}]$-module. This isomorphism is given by

$$|(0)\rangle_{w^{-1}} \otimes |(0)\rangle_w \mapsto [1]_{w^{-1}} \otimes [1]_w.$$

Set $\mathcal{S} := \{c^\lambda_{f_\lambda, v_\lambda}\}_{\lambda \in P_+}$ (f_λ, v_λ are highest weight vectors of $V(\lambda)^*_{\mathrm{gr}}, V(\lambda)$ respectively with $\langle f_\lambda, v_\lambda \rangle = 1$). Then the action of $A_q[\mathfrak{g}]$ on $\tilde{V}_w$ is extended to $A_q[\mathfrak{g}][\mathcal{S}^{-1}]$.

In particular, the $A_q[\mathfrak{g}]$-module $\tilde{V}_w$ is indecomposable, generated by $|(0)\rangle_{w^{-1}} \otimes |(0)\rangle_w$, and decomposed into finite dimensional weight spaces with respect to the action of $\mathcal{S}$. Moreover any nonzero vector generates an infinite dimensional submodule (if $w \neq e$).

Remark

The isomorphism $\tilde{\Psi}^w_{\mathrm{KOY}} : \tilde{V}_w \to U_q^-/U_q^-(w)^\perp \otimes U_q^+/U_q^+(w)^\perp$ in the theorem is NOT equal to $|\mathbf{c}_{\mathrm{rev}}\rangle_{\mathbf{i}_{\mathrm{rev}}} \otimes |\mathbf{c}'\rangle_{\mathbf{i}} \mapsto [F_{\mathbf{i}}^{\mathbf{c}}]_w \otimes [E_{\mathbf{i}}^{\mathbf{c}}]_w$. Indeed

$$\tilde{\Psi}^w_{\mathrm{KOY}}(|\mathbf{c}_{\mathrm{rev}}\rangle_{\mathbf{i}_{\mathrm{rev}}} \otimes |\mathbf{c}'\rangle_{\mathbf{i}'}) = [F_{\mathbf{i}}^{\mathbf{c}}]_w \otimes [E_{\mathbf{i}'}^{\mathbf{c}'}]_w + \sum_{\substack{Y \in U^-, X \in U^+ \ \text{homogeneous} \\ \mathrm{wt}\, Y > \mathrm{wt}\, F_{\mathbf{i}}^{\mathbf{c}}, \mathrm{wt}\, X < \mathrm{wt}\, E_{\mathbf{i}'}^{\mathbf{c}'}}} [Y]_w \otimes [X]_w.$$

We can compute this difference in the case when $\mathfrak{g}$ is of finite type and w is the longest element w_0.

Example

Let $\mathfrak{g} = \mathfrak{sl}_2$. Then

$$\tilde{\Psi}_{\mathrm{KOY}}(|2\rangle \otimes |1\rangle) = F^{(2)} \otimes E + \frac{1}{q - q^3} F \otimes 1.$$

Relation to the Drinfeld double

From now on we assume that $\mathfrak{g}$ is of finite type.
Let U_q' be a variant of the quantized enveloping algebra whose Cartan part is the group algebra of the root lattice ($\subset P$).
Then we can consider the Drinfeld double $A_q[\mathfrak{g}] \bowtie U_q'$ of $A_q[\mathfrak{g}]$ and U_q'. (The $\mathbb{Q}(q)$-algebra $A_q[\mathfrak{g}] \bowtie U_q'$ contains $A_q[\mathfrak{g}]$ and U_q' as subalgebras.)
Then it is known that $\exists$ an embedding $A_q[\mathfrak{g}] \bowtie U_q' \to \check{U}_q \otimes \check{U}_q$ of algebras, whose restriction to $A_q[\mathfrak{g}]$ coincides with Ω.
$\rightsquigarrow$ If the action of $\check{U}_q^{\leq 0} \otimes \check{U}_q^{\geq 0}$ on $U_q^-/U_q^-(w)^\perp \otimes U_q^+/U_q^+(w)^\perp$ extends to the $\check{U}_q \otimes \check{U}_q$-module structure, the $A_q[\mathfrak{g}]$-module structure on $\tilde{V}_w$ extends to the $A_q[\mathfrak{g}] \bowtie U_q'$-module structure!

Relation to the Drinfeld double

Theorem (O-)

Let J be a subset of I and W_J the subgroup of W generated by $\{s_j\}_{j\in J}$. Write the longest element of W (resp. W_J) as w_0 (resp. $w_{J,0}$). Then the $A_q[\mathfrak{g}]$-module structure on $\tilde{V}_{w_0 w_{J,0}}$ can be extended to the $A_q[\mathfrak{g}] \bowtie U'_q$-module structure.

In other words, when $w = w_0 w_{J,0}$, the $A_q[\mathfrak{g}]$-module $\tilde{V}_{w_0 w_{J,0}}$ admits a "compatible U'_q-action".
$\rightsquigarrow$ Can we "understand" the modules of this type conceptually...?
(using quantum flag manifolds...?)