Representations of quantized function algebras and the transition matrices from Canonical bases to PBW bases

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Quantized enveloping algebras

- $\mathfrak{g}$ a finite dimensional complex simple Lie algebra
- $U_q(\mathfrak{g}) = \langle E_i, F_i, K_i^{\pm 1} (i \in I) |$ "usual" relations $\rangle_{\mathbb{Q}(q)\text{-algebra}}$
  (ex. $K_i E_j = q^{(\alpha_i, \alpha_j)} K_i E_j$, q-Serre relations, $\dots$)
  the quantized enveloping algebra$/\mathbb{Q}(q)$ (a q-analogue of $U(\mathfrak{g})$)
- $U_q(\mathfrak{n}^-) = \langle F_i | i \in I \rangle_{\mathbb{Q}(q)\text{-algebra}} \subset U_q(\mathfrak{g})$

The quantized enveloping algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure. In particular, its coalgebra structure is defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,\ \Delta(K_i) = K_i \otimes K_i,$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \varepsilon(K_i^{\pm 1}) = 1.$$

Set $X_i^{(n)} := \dfrac{X_i^n}{[n]_i!}$ where $[n]_i! := \displaystyle\prod_{s=1}^{n} \frac{q_i^s - q_i^{-s}}{q_i - q_i^{-1}}$ and $q_i := q^{(\alpha_i, \alpha_i)/2}$.
$(X = E, F)$

PBW bases

Let $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ be a reduced word of the longest element w_0 of the Weyl group W. (i.e. $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ $N :=$ the length of w_0.)

Definition (The PBW bases)

The vectors

$$F_{\mathbf{i}}^{\mathbf{c}} := F_{i_1}^{(c_1)} T''_{i_1,1}(F_{i_2}^{(c_2)}) \cdots T''_{i_1,1} T''_{i_2,1} \cdots T''_{i_{N-1},1}(F_{i_N}^{(c_N)})$$

$\left(\mathbf{c} = (c_1, c_2, \ldots, c_N) \in (\mathbb{Z}_{\geqq 0})^N\right)$ forms a basis of $U_q(\mathfrak{n}^-)$. Here, $T''_{i,1}$ is a q-analogue of the action of the braid group. (We follow the notation of Lusztig's textbook.)

Note that this is an explicit basis of $U_q(\mathfrak{n}^-)$.
(ex. $\mathfrak{g} = \mathfrak{sl}_3, \mathbf{i} = (1,2,1), F_{\mathbf{i}}^{(1,1,2)} = F_1(F_2 F_1 - q F_1 F_2) F_2^{(2)}$.)

Canonical bases

Let $\mathbf{i}$ be a reduced word of w_0. Then, there uniquely exists a basis $\{ G^{\mathbf{c}} \}_{\mathbf{c}}$ of $U_q(\mathfrak{n}^-)$ such that

- $\overline{G^{\mathbf{c}}} = G^{\mathbf{c}}$, where $\overline{(\cdot)} \circlearrowright U_q(\mathfrak{n}^-)$ given by $F_i \mapsto F_i, q \mapsto q^{-1}$
- $G^{\mathbf{c}} = F_{\mathbf{i}}^{\mathbf{c}} + \sum_{\mathbf{d} > \mathbf{c}} {}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}} F_{\mathbf{i}}^{\mathbf{d}}$ with ${}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}} \in q\mathbb{Z}[q]$.

We consider the lexicographic order on $(\mathbb{Z}_{\geqq 0})^N$.
(ex. $\mathfrak{g}, \mathbf{i}$ as above, $G^{(0,1,0)} = F_2F_1 = (F_2F_1 - qF_1F_2) + qF_1F_2$.)

Definition (The canonical basis)

We call $\{ G^{\mathbf{c}} \}_{\mathbf{c}}$ the canonical basis of $U_q(\mathfrak{n}^-)$.

Remark

In fact, the canonical basis does not depend on the choice of $\mathbf{i}$. (The data $\mathbf{c}$ depend on $\mathbf{i}$.)
In general, this is a not explicit basis despite having “nice” properties.

We investigate ${}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}}$'s via the representation theory of the quantized

Quantized function algebras

The dual space $U_q(\mathfrak{g})^*$ of $U_q(\mathfrak{g})$ has a $\mathbb{Q}(q)$-algebra structure induced from the coalgebra structure of $U_q(\mathfrak{g})$.

Definition (The quantized function algebra)

The quantized function algebra $\mathbb{Q}_q[G]$ is a subalgebra of $U_q(\mathfrak{g})^*$ generated (in fact, spanned) by the matrix coefficients

$$c^{\lambda}_{f,v} := (u \mapsto \langle f, u.v \rangle),$$

here,

- $\lambda \in P_+ (=$ the set of dominant integral weight),
- $V(\lambda)$ the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight λ,
- $f \in V(\lambda)^*, v \in V(\lambda)$.

Hopf algebra structure of QFAs

The algebra $\mathbb{Q}_q[G]$ has a Hopf algebra structure induced from the one of $U_q(\mathfrak{g})$ and a left and right $U_q(\mathfrak{g})$-algebra structure. For example,

- $c^{\lambda}_{f,v} c^{\lambda'}_{f',v'} = c^{V(\lambda)\otimes V(\lambda')}_{f\otimes f',v\otimes v'}$
- $\Delta(c^{\lambda}_{f,v}) = \sum_j c^{\lambda}_{f,v_j} \otimes c^{\lambda}_{f_j,v}$
  $\{v_j\}_j$ a basis of $V(\lambda)$, $\{f_j\}_j$ the dual basis of $V(\lambda)^*$.

Representations of QFAs

The QFA $\mathbb{Q}_q[G]$ is a quantum analogue of the algebra of regular functions on G. (G the connected simply connected simple complex algebraic group whose Lie algebra is $\mathfrak{g}$.)
However, the algebra $\mathbb{Q}_q[G]$ has infinite dimensional irreducible modules:

$$\mathbb{Q}_q[G] \twoheadrightarrow \mathbb{Q}_{q_i}[SL_2] \curvearrowright V_i := \bigoplus_{m \in \mathbb{Z}_{\geqq 0}} \mathbb{Q}(q)|m\rangle_i.$$

(dual to $U_{q_i}(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{g})$.)

Representations of QFAs

Theorem (Soibelman ('90))

Let $w \in W$*. Then, for any reduced expression* $w = s_{i_1} \cdots s_{i_l}$*, the* $\mathbb{Q}_q[G]$*-module* $V_{i_1} \otimes \cdots \otimes V_{i_l}$ *is irreducible and its isomorphism class does not depend on the choice of the reduced expressions. Moreover, if* $\mathbf{i} = (i_1, \ldots, i_l)$ *and* $\mathbf{j} = (j_1, \ldots, j_l)$ *are reduced words of* w*, then there is a* $\mathbb{Q}_q[G]$*-module isomorphism* $\Theta_{\mathbf{j},\mathbf{i}} : V_{i_1} \otimes \cdots \otimes V_{i_l} \to V_{j_1} \otimes \cdots \otimes V_{j_l}$ *given by*

$$|(0)\rangle_{\mathbf{i}} \mapsto |(0)\rangle_{\mathbf{j}},$$

where $|\mathbf{c}\rangle_{\mathbf{k}} := |c_1\rangle_{k_1} \otimes \cdots \otimes |c_l\rangle_{k_l}$ $(\mathbf{k} = \mathbf{i}, \mathbf{j})$*.*

Hence, we denote this module by V_w. (i.e. $V_{i_1} \otimes \cdots \otimes V_{i_l}$ and $V_{j_1} \otimes \cdots \otimes V_{j_l}$ are identified via $\Theta_{\mathbf{j},\mathbf{i}}$.)

KOY's result

Theorem (Kuniba-Okado-Yamada ('13))

Let $\mathbf{i}$, $\mathbf{j}$ *be reduced words of* w_0.
Define the $\mathbb{Q}(q)$*-linear isomorphism* $\Phi_{\mathbf{k}} : U_q(\mathfrak{n}^-) \to V_{w_0}$ *by*
$F_{\mathbf{k}}^{\mathbf{c}} \mapsto |\mathbf{c}\rangle_{\mathbf{k}}$. $(\mathbf{k} = \mathbf{i}, \mathbf{j})$
Then, we have

$$\Phi_{\mathbf{j}} = \Theta_{\mathbf{j},\mathbf{i}} \circ \Phi_{\mathbf{i}}.$$

This theorem says that the map $\Phi_{\mathbf{i}}$ does not depend on the choice of $\mathbf{i}$. (denoted by Φ_{KOY}.)

Strategy

Our approach to the coefficients ${}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}}$:

(Recall that $G^{\mathbf{c}} = F^{\mathbf{c}}_{\mathbf{i}} + \sum_{\mathbf{d}>\mathbf{c}} {}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}} F^{\mathbf{d}}_{\mathbf{i}}$.)

(I) Find an element $c \in \mathbb{Q}_q[G]$ such that $\Phi^{-1}_{\mathrm{KOY}}(c.|(0)\rangle_{\mathbf{i}}) \in U_q(\mathfrak{n}^-)$ "corresponds" to $G^{\mathbf{c}}$.

(II) Compute $c.|(0)\rangle_{\mathbf{i}} (= \Delta^{(N)}(c).|0\rangle_{i_1} \otimes \cdots \otimes |0\rangle_{i_N})$ directly.

In fact, (I) is easy by using "KOY's conjecture" proved by Saito ('14) and Tanisaki ('14).

Notation

Notation

Fix a highest weight vector $v_\lambda \in V(\lambda)$.
Consider the nondegenerate symmetric bilinear form
$(\ ,\ ) : V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ given by

$$(v_\lambda, v_\lambda) = 1, (E_i.u, v) = (u, F_i.v), (K_i.u, v) = (u, K_i.v).$$

For $v \in V(\lambda)$, define $v^* \in V(\lambda)^*$ by $u \mapsto (v, u)$.

Results 1

Fix $\mathbf{c} \in (\mathbb{Z}_{\geqq 0})^N$ and a reduced word $\mathbf{i}$ of w_0.
Take a sufficiently large positive integer L. In particular, ${}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}} \in q^{L-1}\mathbb{Z}[q^{-1}]$ for all $\mathbf{d}$. (This is NOT a sufficient condition of L.)
Set $\lambda := 2(N+1)L\rho$. Here, $\rho :=$ the Weyl vector.

Theorem (O-)

We have

$$\Phi^{-1}_{\mathrm{KOY}}(c^{\lambda}_{(G^{\mathbf{c}}.v_\lambda)^*, v_{w_0\lambda}}.|(0)\rangle_{\mathbf{i}}) = G^{\mathbf{c}} + q^L \sum_{\mathbf{d}\in(\mathbb{Z}_{\geqq 0})^N} \eta_{\mathbf{d}} F^{\mathbf{d}}_{\mathbf{i}} \text{ with } \eta_{\mathbf{d}} \in \mathbb{Z}[q].$$

Here, $v_{w_0\lambda}$ is the lowest weight canonical basis element of $V(\lambda)$.

Remark that $(\mathrm{RHS}) = \sum_{\mathbf{d}} ({}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}} + q^L \eta_{\mathbf{d}}) F^{\mathbf{d}}_{\mathbf{i}}$. Set $G^{\mathbf{c}}.v_\lambda := G^{\mathrm{low}}_\lambda(b_0)$.

Results 2

On the other hand, we have

$$c^{\lambda}_{G^{\mathrm{low}}_{\lambda}(b_0)^*, v_{w_0\lambda}}.|(0)\rangle_{\mathbf{i}} = \sum_{b_1,\dots,b_{N-1}\in B(\lambda)} c^{\lambda}_{G^{\mathrm{low}}_{\lambda}(b_0)^*, G^{\mathrm{up}}_{\lambda}(b_1)}.|0\rangle_{i_1} \otimes c^{\lambda}_{G^{\mathrm{low}}_{\lambda}(b_1)^*, G^{\mathrm{up}}_{\lambda}(b_2)}.|0\rangle_{i_2} \otimes \cdots \otimes c^{\lambda}_{G^{\mathrm{low}}_{\lambda}(b_{N-1})^*, v_{w_0\lambda}}.|0\rangle_{i_N}. \qquad (\spadesuit)$$

Here, $\left\{ G^{\mathrm{low/up}}_{\lambda}(b) \right\}_{b\in B(\lambda)}$ is the canonical/dual canonical basis of $V(\lambda)$.

Results 3

Lemma

In any non-zero summand of the right-hand side of (♠), *for all* k,
$c^{\lambda}_{G^{\mathrm{low}}_{\lambda}(b_{k-1})^{*}, G^{\mathrm{up}}_{\lambda}(b_k)}.|0\rangle_{i_k} = p_k |d_k\rangle_{i_k}$ *with*
$d_k := -\frac{1}{2}\langle \mathrm{wt}\, b_{k-1} + \mathrm{wt}\, b_k, \alpha^{\vee}_{i_k}\rangle$ *and* $p_k \in q^{-L}\mathbb{Z}[q]$.

Hence, we may ignore the degree $\geqq NL$ part of the Laurent polynomial p_k for any k when calculating ${}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}}$.

Results 4

(Recall that $c^{\lambda}_{G^{\mathrm{low}}_{\lambda}(b_{k-1})^{*}, G^{\mathrm{up}}_{\lambda}(b_k)}.|0\rangle_{i_k} = p_k|d_k\rangle_{i_k}$.)

Theorem (O-)

In any summand of the right-hand side of (♠) which contributes to ${}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}}$,

(i) $G^{\mathrm{low}}_{\lambda}(b_k) \in U_q(\mathfrak{n}^-).v_{s_{i_k}\cdots s_{i_1}\lambda}$ $(k = 0, \dots, N)$, *and*

(ii) *the degree* $< NL$ *part of* p_k *is of the form* $q_{i_k}^{\frac{1}{2}d_k(d_k-1)} d^{i_k,d_k}_{\mathbf{s}_k,\mathbf{t}}$ *for some* $\mathbf{t} \in (\mathbb{Z}_{\geqq 0})^N$. $(k = 1, \dots, l(w_0))$

Here, $G^{\mathbf{s}_k}.v_{s_{i_{k-1}}\cdots s_{i_1}\lambda} = G^{\mathrm{low}}_{\lambda}(b_{k-1})$ *and* $(e'_{i_k})^{d_k}(G^{\mathbf{s}_k}) = \sum_{\mathbf{t}} d^{i_k,d_k}_{\mathbf{s}_k,\mathbf{t}} G^{\mathbf{t}}$.
(e'_{i_k} *"the* q*-derivation")*

Remark 1

In fact, this "calculation procedure" coincides with the one using the symmetric bilinear form $(\ ,\ )_{-}$ on $U_q(\mathfrak{n}^-)$ such that

$$(1,1)_{-} = 1 \text{ and } (F_i x, y)_{-} = \frac{1}{1-q_i^2}(x, e_i'(y))_{-}.$$

<u>Sketch</u> We have

$$_{\mathbf{i}}\zeta_{\mathbf{d}}^{\mathbf{c}} = (G^{\mathbf{c}}, \tilde{F}_{\mathbf{i}}^{\mathbf{d}})_{-},$$

where $\tilde{F}_{\mathbf{i}}^{\mathbf{d}}$ is the dual PBW basis.
Calculate the right-hand side using the fact:
For $x, y \in \operatorname{Ker} e_i' (\subset U_q(\mathfrak{n}^-))$, we have

$$(x,y) = ((T''_{i,1})^{-1}(x), (T''_{i,1})^{-1}(y)).$$

"Eliminate $T''_{i,1}$" $\leftrightarrow$ "Go to subsequent Demazure modules"

Remark 2

The following property of canonical bases is technically important:

Proposition (Similarity of the structure constants)

We set

$$F_i^{(p)} G^{\mathrm{low}}(b) = \sum_{\tilde{b} \in B(\infty)} c^{\tilde{b}}_{-pi,b} G^{\mathrm{low}}(\tilde{b}),$$

$$(e_i')^p (G^{\mathrm{low}}(b)) = \sum_{\tilde{b} \in B(\infty)} d^{i,p}_{b,\tilde{b}} G^{\mathrm{low}}(\tilde{b}).$$

Then, for any $b, \hat{b} \in B(\infty), i \in I$ *and* $p \in \mathbb{Z}_{\geqq 0}$, *we have*

$$\left(c^{\hat{b}}_{-pi,b} \right)_{<-\Delta_i(d-1)p} = \left(q_i^{\frac{1}{2}d(d-1)} \begin{bmatrix} \varepsilon_i(\hat{b}) \\ p \end{bmatrix}_i d^{i,d}_{b, \tilde{e}_i^{\varepsilon_i(\hat{b})} \hat{b}} \right)_{<-\Delta_i(d-1)p},$$

where $\Delta_i = (\alpha_i, \alpha_i)/2$ *and* $d := \varepsilon_i(\hat{b}) - p$.

Ref: arXiv1501.01416v2 (Slides: http://www.ms.u-tokyo.ac.jp/˜oya)