Twist automorphisms on quantum unipotent cells and the Chamber Ansatz

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Part of this work is joint with Yoshiyuki Kimura

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Aims

Aims of this talk:

- Establish a quantum analogue of the Chamber Ansatz
  - Relate Feigin homomorphisms to quantum cluster structures
  - Explicit description of quantum twist automorphisms
- The compatibility between quantum twist automorphisms and quantum cluster structures
- Partial results of the periodicity of quantum twist automorphisms

Original story ($q = 1$): Consider the following torus embedding;

$$\begin{array}{cccc} y_{\boldsymbol{i}}\colon & (\mathbb{C}^\times)^\ell & \to & N_-^w := N_- \cap B_+ w B_+ \\ & \Cup & & \Cup \\ & (t_1, \dots, t_\ell) & \longmapsto & \exp(t_1 F_{i_1}) \cdots \exp(t_\ell F_{i_\ell}). \end{array}$$

Here $\boldsymbol{i}$ is a reduced word of w and N_-^w is called a unipotent cell. This gives a birational morphism from $\mathbb{C}^\ell$ to a Schubert variety X_w.
By the way, the restriction $y_{\boldsymbol{i}}\,|_{(\mathbb{R}_{>0})^\ell}$ gives a bijection between $(\mathbb{R}_{>0})^\ell$ and "totally positive elements" in N_-^w [Lusztig].

Problem

Describe the inverse birational morphism $y_{\boldsymbol{i}}^{-1}$.

Berenstein, Fomin, Zelevinsky (1996, 1997) give formulae for $y_{\boldsymbol{i}}^{-1}$, and the resulting substitutions are called "the Chamber Ansatz". The key tool is a twist automorphism $\eta_w\colon N_-^w \to N_-^w$.

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0 = s_1 s_2 s_1$, $\boldsymbol{i} = (1, 2, 1)$.

$$N_-^{w_0} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix} \middle| x_{31} \neq 0, x_{21}x_{32} - x_{31} \neq 0 \right\}.$$

Note that $x_{21}x_{32} - x_{31}$ is the minor corresponding to the row set $\{2, 3\}$ and the column set $\{1, 2\}$. (Such minor will be denoted by $\Delta_{23,12}$.)

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1)$, $N_-^{w_0} = \{\Delta_{3,1} \neq 0, \Delta_{32,12} \neq 0\}$.

$$y_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad y_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}$$

$$y_{\boldsymbol{i}}(t_1, t_2, t_3) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{pmatrix}$$

Then, for $X = \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix}$, we have

$$t_1 = \frac{x_{21}x_{32} - x_{31}}{x_{32}} \qquad t_2 = x_{32} \qquad t_3 = \frac{x_{31}}{x_{32}}.$$

$\rightsquigarrow$ The Chamber Ansatz (later) gives the general formulae!

Introduction (2)

There are known q-analogues $\mathbf{A}_q[N_-^w]$, $\Phi_{\boldsymbol{i}}$ and $\eta_{w,q}$ of $\mathbb{C}[N_-^w]$, $y_{\boldsymbol{i}}$ and η_w^*. The q-analogue $\Phi_{\boldsymbol{i}}$ of $y_{\boldsymbol{i}}$ is called a Feigin homomorphism.

Theorem (O.)

The Chamber Ansatz formulae also hold in quantum settings by using quantum twist automorphisms constructed by Kimura and the author.

By the work of Geiss-Leclerc-Schröer and Goodearl-Yakimov, it is known that there exist many other "embeddings of quantum tori into the quantum unipotent cell" $\mathbf{A}_q[N_-^w]$ as a consequence of their quantum cluster algebra structures.
The relation between these embedding and the embedding $\Phi_{\boldsymbol{i}}$ are given by the quantum Chamber Ansatz formulae.

The Chamber Ansatz $(q = 1)$

Let

- $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ triangular decomposition (fixed),
- $\{E_i, F_i, H_i \mid i \in I\}$ Chevalley generators of $\mathfrak{g}$, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix (i.e. $[H_i, E_j] = a_{ij} E_j, \dots$),
- G connected simply connected algebraic group (over $\mathbb{C}$) with $\mathrm{Lie}\, G = \mathfrak{g}$,
- N_-, H, N_+ closed subgroups of G such that $\mathrm{Lie}\, N_- = \mathfrak{n}^-$, $\mathrm{Lie}\, H = \mathfrak{h}$, $\mathrm{Lie}\, N_+ = \mathfrak{n}^+$,
- $B_- := N_- H$, $B_+ := H N_+$ Borel subgroups,
- $x_i(t) = \exp(tE_i)$, $y_i(t) = \exp(tF_i)$ 1-parameter subgroups corresponding to E_i, F_i,
- $W := N_G(H)/H$ Weyl group, e its unit, $\{s_i \mid i \in I\}$ simple reflections, $\ell(w)$ the length of $w \in W$,

The Chamber Ansatz $(q = 1)$

Let $\mathfrak{g}$, G, $N_{\pm}$, H, $B_{\pm}$, $x_i(t)$, $y_i(t)$, W standard notation.

- $I(w) := \{(i_1, \dots, i_{\ell(w)}) \in I^{\ell(w)} \mid w = s_{i_1} \cdots s_{i_{\ell(w)}}\}$ the set of reduced words of $w \in W$,
- $\overline{s}_i := x_i(-1)y_i(1)x_i(-1)$, $\overline{w} := \overline{s}_{i_1} \cdots \overline{s}_{i_\ell}$, $(i_1, \dots, i_\ell) \in I(w)$. In fact, $\overline{w}$ does not depend on the choice of $(i_1, \dots, i_\ell) \in I(w)$.
- $\{\varpi_i\}_{i \in I} \subset \mathrm{Hom}_{\mathrm{alg.grp.}}(H, \mathbb{C}^{\times})$ fundamental weights.
- $G_0 := N_- H N_+$, and $g = [g]_-[g]_0[g]_+$ $(g \in G_0)$ the corresponding decomposition.

The Chamber Ansatz $(q = 1)$

Let $\mathfrak{g}$, G, $N_\pm$, H, $B_\pm$, $x_i(t)$, $y_i(t)$, W, $I(w)$, $\overline{w}$, ϖ_i standard notation. Set $G_0 := N_- H N_+$, $g = [g]_-[g]_0[g]_+$ $(g \in G_0)$.

Definition (Generalized minors)

For $i \in I$, denote by $\Delta_{\varpi_i,\varpi_i}$ the regular function on G whose restriction to the open dense set G_0 is given by

$$\Delta_{\varpi_i,\varpi_i}(g) := \varpi_i([g]_0)$$

For $w_1, w_2 \in W$, define $\Delta_{w_1\varpi_i,w_2\varpi_i} \in \mathbb{C}[G]$ by

$$\Delta_{w_1\varpi_i,w_2\varpi_i}(g) = \Delta_{\varpi_i,\varpi_i}(\overline{w_1}^{-1} g \overline{w_2})$$

These elements are called *generalized minors*.

The Chamber Ansatz $(q = 1)$ (2)

For $w \in W$, set $N_-^w := N_- \cap B_+ \bar{w} B_+$ unipotent cell.

Fact ([Berenstein, Fomin, Zelevinsky])

There is a biregular morphism $\eta_w \colon N_-^w \to N_-^w$ given by

$$\eta_w(z) := [z^T \overline{w}]_-.$$

This is called a twist automorphism.

Recall the map

$$\begin{array}{cccc} y_{\boldsymbol{i}} \colon & (\mathbb{C}^\times)^\ell & \to & N_-^w \\ & \Downarrow & & \Downarrow \\ & (t_1, \dots, t_\ell) & \longmapsto & y_{i_1}(t_1) \cdots y_{i_\ell}(t_\ell). \end{array}$$

Here $\boldsymbol{i} = (i_1, \dots i_\ell) \in I(w)$.

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There is a biregular morphism $\eta_w \colon N_-^w \to N_-^w$ *given by*

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This is called a twist automorphism.

Theorem (Berenstein, Fomin, Zelevinsky)

Let $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$*. For* $m \in \{1, \ldots, \ell\}$*, set* $w_{\leq m} := s_{i_1} \cdots s_{i_m}$*. Set* $y = y_{\boldsymbol{i}}(t_1, \ldots, t_\ell)$*. Then, for* $k \in \{1, \ldots, \ell\}$*,*

$$t_k = \frac{\prod_{j \in I \setminus \{i_k\}} \Delta_{w_{\leq k}\varpi_j, \varpi_j}(\eta_w^{-1}(y))^{-a_{j,i_k}}}{\Delta_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}}(\eta_w^{-1}(y)) \Delta_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}(\eta_w^{-1}(y))}.$$

These formulae are called the Chamber Ansatz.

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1)$, $N_-^{w_0} = \{\Delta_{3,1} \neq 0, \Delta_{32,12} \neq 0\}$. Recall

$$t_1 = \frac{x_{21}x_{32} - x_{31}}{x_{32}} \qquad t_2 = x_{32} \qquad t_3 = \frac{x_{31}}{x_{32}}.$$

The twist automorphism η_{w_0} is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ x_{32}/x_{31} & 1 & 0 \\ 1/x_{31} & x_{21}/(x_{21}x_{32} - x_{31}) & 1 \end{pmatrix},$$

and $\eta_{w_0}^{-1}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ x_{32}/(x_{21}x_{32} - x_{31}) & 1 & 0 \\ 1/x_{31} & x_{21}/x_{31} & 1 \end{pmatrix}.$$

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1)$, $N_-^{w_0} = \{\Delta_{3,1} \neq 0, \Delta_{32,12} \neq 0\}$. Recall

$$t_1 = \frac{x_{21}x_{32} - x_{31}}{x_{32}} \qquad t_2 = x_{32} \qquad t_3 = \frac{x_{31}}{x_{32}}.$$

$$\eta_{w_0}^{-1} \colon X = \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ x_{32}/(x_{21}x_{32} - x_{31}) & 1 & 0 \\ 1/x_{31} & x_{21}/x_{31} & 1 \end{pmatrix}.$$

Therefore, we have

$$t_1 = \frac{1}{\Delta_{2,1}(\eta_{w_0}^{-1}(X))} \quad t_2 = \frac{\Delta_{2,1}(\eta_{w_0}^{-1}(X))}{\Delta_{23,12}(\eta_{w_0}^{-1}(X))}$$
$$t_3 = \frac{\Delta_{23,12}(\eta_{w_0}^{-1}(X))}{\Delta_{2,1}(\eta_{w_0}^{-1}(X))\Delta_{3,1}(\eta_{w_0}^{-1}(X))}.$$

q-analogue

From now on, we consider a q-analogue of the story above. In the settings of q-analogues, we do not have "actual spaces" but only have "coordinate algebras". Hence we should consider the situations above in terms of coordinate algebras.

The map $y_{\boldsymbol{i}}^*$ induces an injective algebra homomorphism

$$y_{\boldsymbol{i}}^* \colon \mathbb{C}[N_-^w] \to \mathbb{C}[t_1^{\pm 1}, \dots, t_\ell^{\pm}].$$

The twist automorphism η_w induces the algebra automorphism

$$\eta_w^* \colon \mathbb{C}[N_-^w] \to \mathbb{C}[N_-^w].$$

A q-analogue of the former is known as a Feigin homomorphism and that of the latter is a quantum twist automorphism, constructed by Kimura and the author.

Setup

Notation

Let

- $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ a symmetrizable Kac-Moody Lie algebra($\supset$ finite dimensional simple Lie algebra) over $\mathbb{C}$ with (fixed) triangular decomposition,
- $\{\alpha_i\}_{i \in I}$ the simple roots of $\mathfrak{g}$, $\{h_i\}_{i \in I}$ the simple coroots of $\mathfrak{g}$,
- P a $\mathbb{Z}$-lattice (weight lattice) of $\mathfrak{h}^*$ and $P^* := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \subset \mathfrak{h}$ such that $\{\alpha_i\}_{i \in I} \subset P$ and $\{h_i\}_{i \in I} \subset P^*$,
- $P_+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0$ for all $i \in I\}$. Set $\langle h_i, \varpi_j \rangle = \delta_{ij}$.
- W the Weyl group of $\mathfrak{g}$ ($W \curvearrowright P, P^*$),
- $I(w)$ the set of reduced words of $w \in W$,
- $(-,-) : P \times P \to \mathbb{Q}$ a $\mathbb{Q}$-valued (W-invariant) symmetric $\mathbb{Z}$-bilinear form on P satisfying the following conditions: $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$, $\langle \lambda, h_i \rangle = 2\left(\lambda, \alpha_i\right) / \left(\alpha_i, \alpha_i\right)$ for $i \in I$, $\lambda \in P$.

Quantized enveloping algebra

Definition (Quantized enveloping algebras)

The quantized enveloping algebra $\mathbf{U}_q(:= \mathbf{U}_q(\mathfrak{g}))$ over $\mathbb{Q}(q)$ is the $\mathbb{Q}(q)$-algebra generated by

$$e_i, f_i \ (i \in I), \ q^h \ (h \in P^*),$$

with the following relations:

(i) $q^0 = 1, \ q^h q^{h'} = q^{h+h'}$,

(ii) $q^h e_i = q^{\langle h,\alpha_i \rangle} e_i q^h, \ q^h f_i = q^{-\langle h,\alpha_i \rangle} f_i q^h$,

(iii) $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ where $q_i := q^{\frac{(\alpha_i,\alpha_i)}{2}}$ and $t_i := q^{\frac{(\alpha_i,\alpha_i)}{2} h_i}$,

(iv) $\sum_{k=0}^{1-\langle h_i,\alpha_j \rangle} (-1)^k x_i^{(k)} x_j x_i^{(1-\langle h_i,\alpha_j \rangle - k)} = 0$ for $i \neq j$, $x = e, f$,

where $x_i^{(n)} := x_i^n / [n]_i!$, $[n]_i! := \prod_{k=1}^n (q_i^k - q_i^{-k})/(q_i - q_i^{-1})$.

Quantum unipotent subgroup

Let $\mathbf{U}_q^-$ be the subalgebra of $\mathbf{U}_q$ generated by $\{f_i\}_{i\in I}$ and $\mathbf{U}_{\mathbb{Q}[q^{\pm 1}]}^-$ the $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$ generated by $\{f_i^{(n)}\}_{i\in I, n\in\mathbb{Z}_{\geq 0}}$.

Definition

There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_L\colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$ such that

$$(1,1)_L = 1, \qquad (f_i x, y)_L = \frac{1}{1-q_i^2}(x, e_i'(y))_L.$$

where $e_i'\colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ is the $\mathbb{Q}(q)$-linear map given by

$$e_i'(xy) = e_i'(x)\, y + q_i^{\langle \mathrm{wt}\, x, h_i\rangle} x e_i'(y), \quad e_i'(f_j) = \delta_{ij},$$

for homogeneous elements $x, y \in \mathbf{U}_q^-$.

Quantum unipotent subgroup

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Definition

There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_L \colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$.

Set

$$\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] := \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}_{\mathbb{Q}[q^{\pm 1}]}^-)_L \in \mathbb{Q}[q^{\pm 1}]\}.$$

Then $\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-]$ is a $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$.
Specialization:

$$\mathbf{U}_q^- \supset \mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] \xrightarrow[\mathbb{C}\otimes_{\mathbb{Q}[q^{\pm 1}]}-]{\text{“}q\to 1\text{”}} (\mathbf{U}(\mathfrak{n}^-))^*_{\mathrm{gr}} \simeq \mathbb{C}[N_-].$$

Thus we can regard $\mathbf{U}_q^-$ also as a q-analogue of the coordinate algebra $\mathbb{C}[N_-]$.

Quantum closed unipotent cell

Proposition (Kashiwara)

For $w \in W$ *and* $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$*, set*

$$\mathbf{U}_{q,w}^{-} := \sum_{a_1, \cdots, a_\ell} \mathbb{Q}(q) f_{i_1}^{a_1} \cdots f_{i_\ell}^{a_\ell}.$$

Then the following hold:

(1) *The subspace* $\mathbf{U}_{q,w}^{-}$ *does not depend on the choice of* $\boldsymbol{i} \in I(w)$*.*

(2) *Set* $(\mathbf{U}_{q,w}^{-})^{\perp} := \{x \in \mathbf{U}_q^{-} \mid (x, \mathbf{U}_{q,w}^{-})_L = 0\}$*. Then* $(\mathbf{U}_{q,w}^{-})^{\perp}$ *is a two-sided ideal of* $\mathbf{U}_q^{-}$*.*

Set

$$(\mathbf{U}_{q,w}^{-})^{\perp}_{\mathbb{Q}[q^{\pm 1}]} := \{x \in (\mathbf{U}_{q,w}^{-})^{\perp} \mid (x, \mathbf{U}_{\mathbb{Q}[q^{\pm 1}]}^{-})_L \subset \mathbb{Q}[q^{\pm 1}]\},$$

Quantum closed unipotent cell (2)

Definition (Quantum closed unipotent cell)

For $w \in W$, set

$$\mathbf{A}_q[\overline{N_-^w}] := \mathbf{U}_q^- / (\mathbf{U}_{q,w}^-)^\perp = \mathbb{Q}(q) \otimes_{\mathbb{Q}[q^{\pm 1}]} \left(\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] / (\mathbf{U}_{q,w}^-)^\perp_{\mathbb{Q}[q^{\pm 1}]} \right).$$

This is an algebra, called *a quantum closed unipotent cell*, by the proposition above.

In fact, we have

$$\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[\overline{N_-^w}] := \mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] / (\mathbf{U}_{q,w}^-)^\perp_{\mathbb{Q}[q^{\pm 1}]} \xrightarrow[\mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} -]{\text{“}q \to 1\text{”}} \mathbb{C}[\overline{N_-^w}].$$

Unipotent quantum minors

For $\lambda \in P_+$, denote by $V(\lambda)$ the integrable highest weight $\mathbf{U}_q$-module generated by a highest weight vector u_λ of weight λ.
For $w \in W$ and $\boldsymbol{i} \in I(w)$, set

$$u_{w\lambda} = f_{i_1}^{(\langle h_{i_1}, s_{i_2} \cdots s_{i_\ell} \lambda \rangle)} \cdots f_{i_{\ell-1}}^{(\langle h_{i_{\ell-1}}, s_{i_\ell} \lambda \rangle)} f_{i_\ell}^{(\langle h_{i_\ell}, \lambda \rangle)} . u_\lambda .$$

There exists a unique nondegenerate and symmetric bilinear form $(\ ,\ )_\lambda \colon V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ such that

$$(u_\lambda, u_\lambda)_\lambda = 1 \quad (e_i.u, v)_\lambda = (u, f_i.v)_\lambda \quad (q^h.u, v)_\lambda = (u, q^h.v)_\lambda$$

for $u, v \in V(\lambda)$, $i \in I$ and $h \in P^*$.

Definition (Unipotent quantum minors)

For $\lambda \in P_+$ and $u, v \in V(\lambda)$, define an element $D_{u,v} \in \mathbf{U}_q^-$ by

$$(D_{u,v}, x)_L = (u, x.v)_\lambda \text{ for arbitrary } x \in \mathbf{U}_q^- .$$

For $w_1, w_2 \in W$, write $D_{w_1\lambda, w_2\lambda} := D_{u_{w_1\lambda}, u_{w_2\lambda}}$.

Quantum unipotent cell

Proposition

Let $w \in W$. Then $\underline{\mathcal{D}_w} := q^{\mathbb{Z}}\{\underline{D_{w\lambda,\lambda}}\}_{\lambda \in P_+}$ is an Ore set of $\mathbf{A}_q[\overline{N_-^w}]$ consisting of q-central elements.

Definition (Quantum unipotent cells)

For $w \in W$, we can consider the algebras of fractions

$$\mathbf{A}_q[N_-^w] := \mathbf{A}_q[\overline{N_-^w}][\underline{\mathcal{D}_w^{-1}}]$$

by the proposition above. This algebra is called *a quantum unipotent cell*.

Quantum twist maps

Theorem (Kimura-O)

Let $w \in W$. Then there exists the automorphism of the $\mathbb{Q}(q)$-algebra

$$\eta_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w],$$

given by

$$\underline{D_{u,u_\lambda}} \mapsto q^{-(\lambda, \mathrm{wt}\, u - \lambda)} \underline{D_{w\lambda,\lambda}}^{-1} \underline{D_{u_{w\lambda},u}}$$

for all $\lambda \in P_+$ and weight vectors $u \in V(\lambda)$.

We call $\eta_{w,q}$ a quantum twist automorphism.

Feigin homomorphisms

Definition (Feigin homomorphisms)

Let $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I^\ell$. The Laurent q-polynomial algebra $\mathcal{L}_{\boldsymbol{i}}$ is the unital associative $\mathbb{Q}(q)$-algebra generated by $t_1^{\pm 1}, \ldots, t_\ell^{\pm 1}$ subject to the relations;

$$t_j t_k = q^{(\alpha_{i_j}, \alpha_{i_k})} t_k t_j \text{ for } 1 \leq j < k \leq \ell,$$
$$t_k t_k^{-1} = t_k^{-1} t_k = 1 \text{ for } 1 \leq k \leq \ell.$$

Then we can define the $\mathbb{Q}(q)$-linear map $\Phi_{\boldsymbol{i}} \colon \mathbf{U}_q^- \to \mathcal{L}_{\boldsymbol{i}}$ by

$$x \mapsto \sum_{\boldsymbol{a}=(a_1,\ldots,a_\ell)\in\mathbb{Z}_{\geq 0}^\ell} q_{\boldsymbol{i}}(\boldsymbol{a})(x, f_{i_1}^{(a_1)} \cdots f_{i_\ell}^{(a_\ell)})_L t_1^{a_1} \cdots t_\ell^{a_\ell},$$

where $q_{\boldsymbol{i}}(\boldsymbol{a}) := \prod_{k=1}^{\ell} q_{i_k}^{a_k(a_k-1)/2}$. Note that the all but finitely many summands in the right-hand side are zero. The map $\Phi_{\boldsymbol{i}}$ is called *a Feigin homomorphism*.

Feigin homomorphisms (2)

Proposition (Berenstein)

(1) *For* $\boldsymbol{i} \in I^{\ell}$, *the map* $\Phi_{\boldsymbol{i}}$ *is a* $\mathbb{Q}(q)$*-algebra homomorphism.*

(2) *For* $w \in W$ *and* $\boldsymbol{i} \in I(w)$, *we have* $\operatorname{Ker} \Phi_{\boldsymbol{i}} = \left(\mathbf{U}^{-}_{w,q}\right)^{\perp}$.

(3) *For* $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ *and* $\lambda \in P_+$, *we have*

$$\Phi_{\boldsymbol{i}}\left(D_{w\lambda,\lambda}\right) = q_{\boldsymbol{i}}(\boldsymbol{d}) t_1^{d_1} \cdots t_\ell^{d_\ell}$$

where $\boldsymbol{d} = (d_1, \dots, d_\ell)$ *with* $d_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_\ell} \lambda \rangle$.

Hence $\Phi_{\boldsymbol{i}}$ gives rise to an injective algebra homomorphism

$$\Phi_{\boldsymbol{i}} \colon \mathbf{A}_q[N_-^w] \to \mathcal{L}_{\boldsymbol{i}}.$$

The quantum Chamber Ansatz

Theorem (O.)

Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ *and* $k \in \{1, \dots, \ell\}$. *Then*

$$(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})(\underline{D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}}) = \left(\prod_{j=1}^{k} q_{i_j}^{d_j(d_j+1)/2}\right) t_1^{-d_1} t_2^{-d_2} \cdots t_k^{-d_k},$$

where $d_j := \langle h_{i_j}, s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k} \rangle$ $(j = 1, \dots, k)$. *Denote this element by* $D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}} \in \mathcal{L}_{\boldsymbol{i}}$.

Corollary (The quantum Chamber Ansatz)

Let $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. *Then, for* $k \in \{1, \dots, \ell\}$,

$$t_k \simeq (D'^{(\boldsymbol{i})}_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}})^{-1} (D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}})^{-1} \prod_{j \in I \setminus \{i_k\}} (D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_j, \varpi_j})^{-a_{j,i_k}},$$

here the right-hand side is determined up to powers of q.

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1,2,1)$. Write $D_{s_1\pi_1,\pi_1} = D_{2,1}$ etc.. (In type A, the unipotent quantum minors associated with the fundamental representations correspond to the q-analogues of usual minors.)

$$\eta_{w,q}^{-1}(D_{2,1}) = D_{23,12}^{-1}D_{13,12}, \eta_{w,q}^{-1}(D_{23,12}) = qD_{23,12}^{-1}, \eta_{w,q}^{-1}(D_{3,1}) = qD_{3,1}^{-1}.$$

$$\left(\text{cf. } \eta_{w_0}^{-1}(X) = \begin{pmatrix} 1 & 0 & 0 \\ x_{32}/(x_{21}x_{32} - x_{31}) & 1 & 0 \\ 1/x_{31} & x_{21}/x_{31} & 1 \end{pmatrix}.\right)$$

$$D'^{(\boldsymbol{i})}_{2,1} = qt_1^{-1} \qquad D'^{(\boldsymbol{i})}_{23,12} = q^2t_1^{-1}t_2^{-1} \qquad D'^{(\boldsymbol{i})}_{3,1} = q^2t_2^{-1}t_3^{-1}.$$

Hence,

$$t_1 = q(D'^{(\boldsymbol{i})}_{2,1})^{-1} \quad t_2 = q(D'^{(\boldsymbol{i})}_{23,12})^{-1}D'^{(\boldsymbol{i})}_{2,1}$$
$$t_3 = (D'^{(\boldsymbol{i})}_{2,1})^{-1}(D'^{(\boldsymbol{i})}_{3,1})^{-1}D'^{(\boldsymbol{i})}_{23,12}.$$

Quantum cluster algebra

A quantum cluster algebra is a subalgebra of the fraction field $\mathcal{F}$ of a quantum torus $\mathcal{T}_M$, determined by a skew-symmetric bilinear form $\Lambda\colon \mathbb{Z}^\ell \times \mathbb{Z}^\ell \to \mathbb{Z}$, which determines the data of the q-commutativity of the variables.

The initial data (M, Λ_M, B), called *an initial quantum seed*.

- M a toric chart, which indicates a quantum torus $\mathcal{T}_{\Lambda_M}$ (or quantum cluster) inside $\mathcal{F}$
- B a exchange matrix, which governs *mutation*.

The quantum cluster algebra $\mathscr{A}_{q^{\pm 1/2}}(M, \Lambda_M, B)$ is defined as the $\mathbb{Q}[q^{\pm 1/2}]$-subalgebra of $\mathcal{F}$ generated by all quantum clusters obtained by iterated mutations.

Quantum cluster algebra (2)

The following property is known as *the Laurent phenomenon*.

Proposition ([Berenstein-Zelevinsky])

$$\mathscr{A}_{q^{\pm 1/2}}(M, \Lambda_M, B) \subset \bigcap_M \mathcal{T}_M.$$

Geiss-Leclerc-Schröer have realized the quantum unipotent cell $\mathbf{A}_q[N_-^w]$ as the quantum cluster algebras constructed from the representations of preprojective algebras. This is called *the additive categorification* of $\mathbf{A}_q[N_-^w]$. In particular, we have the two kinds of "quantum torus embedding";

$$\mathcal{L}_{\boldsymbol{i}} \xleftarrow{\text{Feigin homomorphism}} \mathbf{A}_q[N_-^w] \xrightarrow{\text{cluster structure}} \mathcal{T}_M$$

We will explain their relations. (In fact, the answer is already given by the quantum Chamber Ansatz!)

Relation with GLS theory

From now on, we assume that $\mathfrak{g}$ is a symmetric Kac-Moody Lie algebra. Let $Q = (Q_0, Q_1, s, t)$ be a corresponding finite quiver without oriented cycles. Denote by Λ the preprojective algebra corresponding to Q, that is,

$$\Lambda := \mathbb{C}\overline{Q}/(\sum_{a \in Q_1}(a^*a - aa^*)),$$

here $\mathbb{C}\overline{Q}$ is the path algebra of the double quiver of Q. For a nilpotent Λ-module X, we can define $\varphi_X \in \mathbb{C}[N_-^w]$ satisfying the following:

$$\varphi_X(y_{\boldsymbol{i}}(t_1, \dots, t_\ell)) = \sum_{\boldsymbol{a}=(a_1,\dots,a_\ell)\in\mathbb{Z}_{\geq 0}^\ell} \chi(\mathcal{F}_{\boldsymbol{i},\boldsymbol{a},X}) t_1^{a_1} \cdots t_\ell^{a_\ell},$$

here $\boldsymbol{i} \in I(w)$, χ denotes the Euler characteristic, and $\mathcal{F}_{\boldsymbol{i},\boldsymbol{a},X}$ is the projective variety of flags $X_\bullet = (X = X_0 \supset X_1 \supset \cdots \supset X_\ell = 0)$ of submodules of X such that $X_{k-1}/X_k \simeq S_{i_k}^{a_k}$ for $1 \leq k \leq \ell$ [Lusztig].

Relation with GLS theory (2)

Buan-Iyama-Reiten-Scott have constructed a 2-Calabi-Yau Frobenius subcategory $\mathcal{C}_w$ of Λ-modules, and Geiß-Leclerc-Schröer have proved that

$$\mathbb{C}[N_-^w] = \mathrm{span}_{\mathbb{C}}\{\varphi_X \mid X \in \mathcal{C}_w\}[\{\varphi_I \mid I \colon \mathcal{C}_w\text{-injective-projective}\}^{-1}].$$

Here an object is projective in $\mathcal{C}_w$ ($\mathcal{C}_w$-projective) if and only if it is injective in $\mathcal{C}_w$ ($\mathcal{C}_w$-injective) since $\mathcal{C}_w$ is Frobenius.
For $X \in \mathcal{C}_w$, denote by $I(X)$ the injective hull of X in $\mathcal{C}_w$, and by $\Omega_w^{-1}(X)$ the cokernel of $X \to I(X)$.

$$0 \to X \to I(X) \to \Omega_w^{-1}(X) \to 0.$$

Relation with GLS theory (3)

Theorem (GLS)

Let $w \in W$. *For* $X \in \mathcal{C}_w$, $\eta_w^*(\varphi_X) = \varphi_{I(X)}^{-1} \varphi_{\Omega_w^{-1}(X)}$.

GLS have also constructed the algebra $\mathbf{A}_q[N^w]$ from $\mathcal{C}_w$ and constructed a q-analogue of φ_M, denoted by Y_M, for every reachable rigid module M. By using the theorem above, we obtain the following:

Theorem (Kimura-O)

Let $w \in W$. *For a reachable rigid module* M,

$$\eta_{w,q}(Y_M) \simeq Y_{I(M)}^{-1} Y_{\Omega_w^{-1}(M)}.$$

This theorem states that *quantum cluster monomials (which admit the inverses at the frozen part) are preserved by the quantum twist automorphism*.

Relation with GLS theory (4)

In GLS's categorification, the initial seed $\mathbb{Y}_{\boldsymbol{i}} := \{Y_{M_{\boldsymbol{i},k}}\}_{k=1,\dots,\ell}$ corresponds to $\{D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}\}_{k=1,\dots,\ell}$. This is noting but the elements appearing in the (quantum) Chamber Ansatz formulae.
By Theorem above, the elements $\mathbb{Y}'_{\boldsymbol{i}} := \{\eta_{w,q}^{-1}(Y_{M_{\boldsymbol{i},k}})\}_{i=1,\dots,\ell}$ are also a cluster (up to frozen variables). Hence, via the quantum Chamber Ansatz formulae,

Calculating the image of the Feigin homomorphism "="
Calculating the cluster expansion with respect to $\mathbb{Y}'_{\boldsymbol{i}}$.

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1)$.

$\eta_{w,q}^{-1}(D_{2,1}) = D_{23,12}^{-1} D_{13,12}, \eta_{w,q}^{-1}(D_{23,12}) = q D_{23,12}^{-1}, \eta_{w,q}^{-1}(D_{3,1}) = q D_{3,1}^{-1}$.

Now $\{D_{2,1}, D_{23,12}, D_{3,1}\}$ is the initial quantum cluster, and $\{D_{13,12}, D_{23,12}, D_{3,1}\}$ is also a quantum cluster.
For example, we have $\Phi_{\boldsymbol{i}}(D_{2,1}) = t_1 + t_3$. Hence,

$$\Phi_{\boldsymbol{i}}(D_{2,1}) = q(D'^{(\boldsymbol{i})}_{2,1})^{-1} + (D'^{(\boldsymbol{i})}_{2,1})^{-1}(D'^{(\boldsymbol{i})}_{3,1})^{-1} D'^{(\boldsymbol{i})}_{23,12}.$$

Therefore,

$$\begin{aligned} D_{2,1} &= q D_{13,12}^{-1} D_{23,12} + D_{13,12}^{-1} D_{23,12} D_{3,1} D_{23,12}^{-1} \\ &= q D_{13,12}^{-1} D_{23,12} + D_{13,12}^{-1} D_{3,1}. \end{aligned}$$

Periodicity

It is known that

$$(\Omega_{w_0}^{-1})^6(M) \simeq M$$

for an indecomposable non-projective-injective module M. This property suggests (and proves in the ADE case) the "6-periodicity" of the specific quantum twist automorphism $\eta_{w_0,q}$. Assume that $\mathfrak{g}$ is finite dimensional, and let w_0 be the longest element of W.

Theorem (Kimura-O.)

For a homogeneous element $x \in \mathbf{A}_q[N_-^{w_0}]$, we have

$$\eta_{w_0,q}^6(x) = q^{(\mathrm{wt}\, x + w_0 \,\mathrm{wt}\, x, \mathrm{wt}\, x)} D_{w_0, -\mathrm{wt}\, x - w_0 \,\mathrm{wt}\, x}\, x.$$

We proved this theorem purely in the algebraic method (not using the categorification). Hence this result is also valid for the non-symmetric case.

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When the action of w_0 on P is given by $\mu \mapsto -\mu$, the theorem above states that $\eta_{w_0,q}^6 = \mathrm{id}$ ("really" periodic). If $\mathfrak{g}$ is simple, then this condition is satisfied in the case that $\mathfrak{g}$ is of type B_n, C_n, D_{2n} for $n \in \mathbb{Z}_{>0}$ and E_7, E_8, F_4, G_2.