Quantum Grothendieck ring isomorphisms for quantum affine algebras of type A and B

Hironori OYA

Université Paris Diderot, IMJ-PRG

Based on a joint work with David HERNANDEZ

Algebraic Lie Theory and Representation Theory 2018
Karuizawa, May 27, 2018

Motivation (1)

Topic : Finite dimensional representations of affine quantum groups

Question 1

Dimensions/q-characters of simple modules ?

- $\exists$ Classification of simple modules [Chari-Pressley 1990's]
  "Highest weight theory"
- However, there are no known closed formulae of their dimensions and q-characters in general. (e.g. $\nexists$ analogue of Weyl-Kac character formulae...)

Question 2

Description of representation rings and their "deformations" ?

- Some (deformed) representation rings are known to be described nicely as (quantum) cluster algebras...

Motivation (2)

Question 1

Dimensions/q-characters of simple modules ?

- $\underline{\text{ADE}}$ case $\exists$ algorithm to compute them ! [Nakajima '04]
  "Kazhdan-Lusztig algorithm"

  The tool is t-deformed q-characters, and the geometric construction (via quiver varieties) of simple modules guarantees this algorithm.
- Arbitrary (untwisted) case [Hernandez '04]
  - $\exists$ t-deformed q-characters, defined algebraically ($\nexists$ geometry for non-symmetric cases)
  - Kazhdan-Lusztig algorithm gives conjectural q-characters of simple modules

However, they are still candidates in non-symmetric cases.

Motivation (3)

Question 2

Description of representation rings and their "deformations" ?

– [Hernandez-Leclerc '10 –, Kang-Kashiwara-Kim-Oh '15, Oh-Suh '16] The category of finite dimensional modules of affine quantum groups has several interesting monoidal subcategories ($\mathcal{C}_{\mathbb{Z}}$, $\mathcal{C}_{\mathbb{Z}}^{-}$, $\mathcal{C}_{\ell}$, $\ell \in \mathbb{Z}$, $\mathcal{C}_{\mathcal{Q}}$ etc.), which are expected to be "monoidal categorifications" of cluster algebras (this fact is indeed proved in many cases).

Motivation (3)

Question 2

Description of representation rings and their "deformations" ?

– $\underline{\mathrm{X} = \mathrm{ADE}\text{ case}}$ Let

- $K_t(\mathcal{C}_{\mathcal{Q},\mathrm{X}_n^{(1)}})$ the t-deformed Grothendieck ring (=quantum Grothendieck ring) of $\mathcal{C}_{\mathcal{Q},\mathrm{X}_n^{(1)}}$ for type $\mathrm{X}_n^{(1)}$
- $\mathcal{A}_v[N_-^{\mathrm{X}_n}]$ the quantized coordinate algebra of the unipotent group of type X_n ($\exists$ quantum cluster algebra structure !)

(Each terminology will be explained later.)

Theorem (Hernandez-Leclerc '15)

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{X}_n^{(1)}}) \simeq \mathcal{A}_v[N_-^{\mathrm{X}_n}], \left\{ \begin{array}{c} (q,t)\text{-characters of} \\ \text{simple modules} \end{array} \right\} \leftrightarrow \textit{dual canonical basis.}$$

Does it also hold in non-symmetric cases ?

Overview of Main results

In this talk, we consider the case of type $\mathrm{B}_n^{(1)}$. Let $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ be the monoidal subcategory $\mathcal{C}_{\mathcal{Q}}$ for type $\mathrm{B}_n^{(1)}$.

Theorem (Hernandez-O.)

$$\begin{array}{ccccc}
K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) & \simeq & \mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] & \overset{\text{[HL]}}{\simeq} & K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_{2n-1}^{(1)}}) \\
\cup & & \cup & & \cup \\
\left\{ \begin{array}{c} (q,t)\text{-}\textit{characters of} \\ \textit{simple modules} \end{array} \right\} & \leftrightarrow & \textit{dual canonical basis} & \overset{\text{[HL]}}{\longleftrightarrow} & \left\{ \begin{array}{c} (q,t)\text{-}\textit{characters of} \\ \textit{simple modules} \end{array} \right\}
\end{array}$$

Remark

There are no known direct relations between the quantum affine algebras of type $\mathrm{B}_n^{(1)}$ and $\mathrm{A}_{2n-1}^{(1)}$ themselves !

Overview of Main results

In this talk, we consider the case of type $\mathrm{B}_n^{(1)}$. Let $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ be the monoidal subcategory $\mathcal{C}_{\mathcal{Q}}$ for type $\mathrm{B}_n^{(1)}$.

Theorem (Hernandez-O.)

$$\begin{array}{ccccc} K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) & \simeq & \mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] & \overset{\text{[HL]}}{\simeq} & K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_{2n-1}^{(1)}}) \\ \cup & & \cup & & \cup \\ \left\{ \begin{array}{c} (q,t)\text{-characters of} \\ \text{simple modules} \end{array} \right\} & \leftrightarrow & \text{dual canonical basis} & \overset{\text{[HL]}}{\longleftrightarrow} & \left\{ \begin{array}{c} (q,t)\text{-characters of} \\ \text{simple modules} \end{array} \right\} \end{array}$$

Kashiwara-Oh established an isomorphism between $K_{t=1}(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}})$ and $\mathbb{C}[N_-^{\mathrm{A}_{2n-1}}]$ by a different method. Combining this result with our theorem above, we obtain the following :

Overview of Main results

Let $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ be the monoidal subcategory $\mathcal{C}_{\mathcal{Q}}$ for type $\mathrm{B}_n^{(1)}$.

Theorem (Hernandez-O.)

$$\begin{array}{ccccc} K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) & \simeq & \mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] & \overset{\text{[HL]}}{\simeq} & K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_{2n-1}^{(1)}}) \\ \cup & & \cup & & \cup \\ \left\{ \begin{array}{c} (q,t)\text{-characters of} \\ \text{simple modules} \end{array} \right\} & \leftrightarrow & \text{dual canonical basis} & \overset{\text{[HL]}}{\longleftrightarrow} & \left\{ \begin{array}{c} (q,t)\text{-characters of} \\ \text{simple modules} \end{array} \right\} \end{array}$$

Kashiwara-Oh established an isomorphism between $K_{t=1}(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}})$ and $\mathbb{C}[N_-^{\mathrm{A}_{2n-1}}]$ by a different method. Combining this result with our theorem above, we obtain the following :

Theorem (Hernandez-O.)

The (q,t)-characters of simple modules in $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ specialize to the corresponding q-characters.

Overview of Main results

Let $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ be the monoidal subcategory $\mathcal{C}_{\mathcal{Q}}$ for type $\mathrm{B}_n^{(1)}$.

Theorem (Hernandez-O.)

$$\begin{array}{ccccc}
K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) & \simeq & \mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] & \overset{\text{[HL]}}{\simeq} & K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_{2n-1}^{(1)}}) \\
\cup & & \cup & & \cup \\
\left\{ \begin{array}{c} (q,t)\text{-characters of} \\ \text{simple modules} \end{array} \right\} & \leftrightarrow & \text{dual canonical basis} & \overset{\text{[HL]}}{\longleftrightarrow} & \left\{ \begin{array}{c} (q,t)\text{-characters of} \\ \text{simple modules} \end{array} \right\}
\end{array}$$

Theorem (Hernandez-O.)

The (q,t)-characters of simple modules in $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ specialize to the corresponding q-characters.

$\rightsquigarrow$ The Kazhdan-Lusztig algorithm gives "correct" answers in $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$!

Quantum affine algebras

Let

- $\mathfrak{g}$ a finite dimensional simple Lie algebra / $\mathbb{C}$
- $\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ its loop algebra $[X \otimes t^m, Y \otimes t^m] = [X, Y] \otimes t^{m+m'}$
- $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ the Drinfeld-Jimbo quantum loop algebra / $\mathbb{C}$ with a parameter $q \in \mathbb{C}^\times$ not a root of unity
  generators : $\{k_i^{\pm 1}, x_{i,r}^{\pm}, h_{i,s} \mid i \in I, r \in \mathbb{Z}, s \in \mathbb{Z} \setminus \{0\}\}$

Properties

- $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ has a Hopf algebra structure.
- $\mathcal{U}_q(\mathfrak{g}) \underset{\text{Hopf alg.}}{\hookrightarrow} \mathcal{U}_q(\mathcal{L}\mathfrak{g}), e_i \mapsto x_{i,0}^+, f_i \mapsto x_{i,0}^-, k_i^{\pm 1} \mapsto k_i^{\pm 1}$.

Let $\mathcal{C}$ be the category of finite-dimensional $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-modules of type 1

(i.e. the eigenvalues of the actions of $\{k_i \mid i \in I\}$ are of the form q^m, $m \in \mathbb{Z}$).

Remark : $\mathcal{C}$ is a non-semisimple abelian $\otimes$-category.

q-characters (1)

Let $V \in \mathcal{C}$. Frenkel-Reshetikhin showed that

$$\{\text{Generalized simultaneous eigenvalues of all } k_i^{\pm 1}, h_{i,s} \curvearrowright V\} \overset{1:1}{\leftrightsquigarrow} \{\text{Laurent monomials } m \text{ in } Y_{i,a}\text{'s } (i \in I, a \in \mathbb{C}^\times)\}$$

$\rightsquigarrow V = \bigoplus_m V_m$, called the ℓ-weight space decomposition.

$Y_{i,a}$ is an "affine analogue" of e^{ϖ_i}, ϖ_i fundamental weight.

Define the q-character of V as

$$\chi_q(V) := \sum_m \dim(V_m) m.$$

Then χ_q defines an injective algebra homomorphism

$$\chi_q \colon K(\mathcal{C}) \to \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^\times] =: \mathcal{Y}_{\mathbb{C}^\times},$$

here $K(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$ [Frenkel-Reshetikhin].

$K(\mathcal{C})$ is commutative. (However sometimes $V \otimes W \not\simeq W \otimes V$ in $\mathcal{C}$.)

q-characters (2)

Set $\mathcal{B}_{\mathbb{C}^\times} := \left\{ \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{m_{i,a}} \mid m_{i,a} \geq 0 \right\} \subset \mathcal{Y}_{\mathbb{C}^\times}$ dominant monomials.

Classification of simple modules [Chari-Pressley]

There is a one-to-one correspondence :

$$\begin{array}{ccc} \{\text{simple modules in } \mathcal{C}\} / \sim & \leftrightarrow & \mathcal{B}_{\mathbb{C}^\times} \\ \Downarrow & & \Downarrow \\ [L(m)] & \leftrightarrow & m \end{array}$$

$\exists$ an "affine analogue" $A_{i,a} \in \mathcal{Y}_{\mathbb{C}^\times}$ of e^{α_i}, α_i simple root.

Type $\mathrm{A}_n^{(1)}$

$$A_{i,a} = Y_{i,aq^{-1}} Y_{i,aq} Y_{i-1,a}^{-1} Y_{i+1,a}^{-1} \ (\leftrightsquigarrow e^{\alpha_i} = e^{2\varpi_i - \varpi_{i-1} - \varpi_{i+1}})$$

$(Y_{0,a} = Y_{n+1,a} := 1,\ e^{\varpi_0} = e^{\varpi_{n+1}} := 1.)$

Set $\mathcal{B}_{\mathbb{C}^\times} := \left\{ \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{m_{i,a}} \mid m_{i,a} \geq 0 \right\} \subset \mathcal{Y}_{\mathbb{C}^\times}$ dominant monomials.

Classification of simple modules [Chari-Pressley]

There is a one-to-one correspondence :

$$\begin{array}{ccc} \{\text{simple modules in } \mathcal{C}\} / \sim & \leftrightarrow & \mathcal{B}_{\mathbb{C}^\times} \\ \Cup & & \Cup \\ [L(m)] & \leftrightarrow & m \end{array}$$

$\exists$ an "affine analogue" $A_{i,a} \in \mathcal{Y}_{\mathbb{C}^\times}$ of e^{α_i}, α_i simple root.

Type $\mathrm{B}_n^{(1)}$

$$A_{i,a} = \begin{cases} Y_{i,aq^{-2}} Y_{i,aq^2} Y_{i-1,a}^{-1} Y_{i+1,a}^{-1} & \text{if } i \leq n-2 \\ Y_{n-1,aq^{-2}} Y_{n-1,aq^2} Y_{n-2,a}^{-1} Y_{n,aq^{-1}}^{-1} Y_{j,aq}^{-1} & \text{if } i = n-1 \\ Y_{n,aq^{-1}} Y_{n,aq} Y_{n-1,a}^{-1} & \text{if } i = n. \end{cases}$$

$(Y_{0,a} := 1)$

q-characters (2)

Set $\mathcal{B}_{\mathbb{C}^\times} := \left\{ \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{m_{i,a}} \mid m_{i,a} \geq 0 \right\} \subset \mathcal{Y}_{\mathbb{C}^\times}$ dominant monomials.

Classification of simple modules [Chari-Pressley]

There is a one-to-one correspondence :

$$\begin{array}{ccc} \{\text{simple modules in } \mathcal{C}\} / \sim & \leftrightarrow & \mathcal{B}_{\mathbb{C}^\times} \\ \Cup & & \Cup \\ [L(m)] & \leftrightarrow & m \end{array}$$

$\exists$ an "affine analogue" $A_{i,a} \in \mathcal{Y}_{\mathbb{C}^\times}$ of e^{α_i}, α_i simple root.

Define the partial ordering on the set of Laurent monomials in $\mathcal{Y}_{\mathbb{C}^\times}$ as

$$m \geq m' \Leftrightarrow m^{-1}m' \text{ is a product of } A_{i,a}^{-1}\text{'s.}$$

Theorem (Frenkel-Mukhin)

$$\chi_q(L(m)) = m + \textit{(sum of terms lower than } m\textit{)}, \ \forall m \in \mathcal{B}_{\mathbb{C}^\times}.$$

q-characters (3)

$\mathcal{C}_\bullet$:=the full subcategory of $\mathcal{C}$ such that

object : V with $\chi_q(V) \in \mathbb{Z}[Y_{i,q^r}^{\pm 1} \mid i \in I, r \in \mathbb{Z}] =: \mathcal{Y}$.

Properties

- $\mathcal{C}_\bullet$ is a (non-semisimple) abelian $\otimes$-subcategory.
- $\mathcal{C} = \bigotimes_{a \in \mathbb{C}^\times / q^{\mathbb{Z}}} (\mathcal{C}_\bullet)_a$ ($(\mathcal{C}_\bullet)_a$ is obtained from $\mathcal{C}_\bullet$ by shift of the spectral parameter by a).

From now on, we always work in $\mathcal{C}_\bullet$, and write

$$Y_{i,r} := Y_{i,q^r} \qquad A_{i,r} := A_{i,q^r} \qquad \mathcal{B} := \mathcal{B}_{\mathbb{C}^\times} \cap \mathcal{Y}.$$

Example

- $\mathfrak{g} = \mathfrak{sl}_2, I = \{1\}$, $\chi_q(L(Y_{1,r})) = Y_{1,r} + Y_{1,r+2}^{-1} = Y_{1,r}(1 + A_{1,r+1}^{-1})$.
- $\mathfrak{g} = \mathfrak{so}_5, I = \{1, 2\}$,

$$\chi_q(L(Y_{1,r})) = Y_{1,r} + Y_{2,r+1}Y_{2,r+3}Y_{1,r+4}^{-1} + Y_{2,r+1}Y_{2,r+5}^{-1} + Y_{1,r+2}Y_{2,r+3}^{-1}Y_{2,r+5}^{-1} + Y_{1,r+6}^{-1}.$$

q-characters (3)

$\mathcal{C}_\bullet$:=the full subcategory of $\mathcal{C}$ such that

object : V with $\chi_q(V) \in \mathbb{Z}[Y_{i,q^r}^{\pm 1} \mid i \in I, r \in \mathbb{Z}] =: \mathcal{Y}$.

Properties

- $\mathcal{C}_\bullet$ is a (non-semisimple) abelian $\otimes$-subcategory.
- $\mathcal{C} = \bigotimes_{a \in \mathbb{C}^\times / q^{\mathbb{Z}}} (\mathcal{C}_\bullet)_a$ ($(\mathcal{C}_\bullet)_a$ is obtained from $\mathcal{C}_\bullet$ by shift of the spectral parameter by a).

From now on, we always work in $\mathcal{C}_\bullet$, and write

$$Y_{i,r} := Y_{i,q^r} \qquad A_{i,r} := A_{i,q^r} \qquad \mathcal{B} := \mathcal{B}_{\mathbb{C}^\times} \cap \mathcal{Y}.$$

For $m = \prod_{i \in I, r \in \mathbb{Z}} Y_{i,r}^{u_{i,r}} \in \mathcal{B}$, a standard module is defined as

$$M(m) := \overrightarrow{\bigotimes_{r \in \mathbb{Z}}} \left(\bigotimes_{i \in I} L(Y_{i,r})^{\otimes u_{i,r}} \right).$$

$\rightsquigarrow$ $\{[L(m)] \mid m \in \mathcal{B}\}$ and $\{[M(m)] \mid m \in \mathcal{B}\}$ are $\mathbb{Z}$-bases of $K(\mathcal{C}_\bullet)$.

Quantum Grothendieck rings (1)

We recall Hernandez's algebraic construction of quantum Grothendieck rings here.

Remark

$\exists$ other (geometric) constructions given by Varagnolo-Vasserot or Nakajima for ADE cases, and all constructions produce equivalent rings in these cases.

Quantum Grothendieck rings (1)

We recall Hernandez's algebraic construction of quantum Grothendieck rings here.
Let $C = (c_{ij})_{i,j \in I}$ be the Cartan matrix of $\mathfrak{g}$, and $D = (\delta_{ij} r_i)_{i,j \in I}$ such that $r_i \in \mathbb{Z}_{>0}$, $\gcd_{i \in I}(r_i) = 1$ and DC is symmetric.
Define $C(z) = (C(z)_{ij})_{i,j \in I}, \widetilde{C}(z) = (\widetilde{C}(z)_{ij})_{i,j \in I}$ (z : indeterminate) by

$$C(z)_{ij} = \begin{cases} z^{r_i} + z^{-r_i} & \text{if } i = j \\ [c_{ij}]_z & \text{if } i \neq j \end{cases}, \qquad \widetilde{C}(z) = C(z)^{-1}.$$

Here $[m]_z := \dfrac{z^m - z^{-m}}{z - z^{-1}}$.
We can regard $\widetilde{C}(z)_{ij}$ as an element of $\mathbb{Z}((z^{-1}))$, and write

$$\widetilde{C}(z)_{ji} = \sum_{r \in \mathbb{Z}} \widetilde{c}_{ji}(r) z^r \in \mathbb{Z}((z^{-1})).$$

Example

If $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ (type B_2), then

$$C(z) = \begin{pmatrix} z^2 + z^{-2} & -1 \\ -z - z^{-1} & z + z^{-1} \end{pmatrix} \qquad \widetilde{C}(z) = \frac{1}{z^3 + z^{-3}} \begin{pmatrix} z + z^{-1} & 1 \\ z + z^{-1} & z^2 + z^{-2} \end{pmatrix}.$$

Hence

$$\widetilde{C}(z)_{11} = \sum_{k \geq 0} (-1)^k (z^{-6k-2} + z^{-6k-4}), \qquad \widetilde{C}(z)_{12} = \sum_{k \geq 0} (-1)^k z^{-6k-3},$$

$$\widetilde{C}(z)_{21} = \sum_{k \geq 0} (-1)^k (z^{-6k-2} + z^{-6k-4}), \qquad \widetilde{C}(z)_{22} = \sum_{k \geq 0} (-1)^k (z^{-6k-1} + z^{-6k-5}).$$

Hence $\widetilde{c}_{ji}(r)$'s are summarized as follows (blanks stand for 0) :

$\widetilde{c}_{j1}(r)$ \ r	-2	-4	-6	-8	-10	-12	-14	-16	$\cdots$
j = 1	1	1		-1	-1		1	1	$\cdots$
j = 2	1	1		-1	-1		1	1	$\cdots$

$\widetilde{c}_{j2}(r)$ \ r	-1	-3	-5	-7	-9	-11	-13	-15	$\cdots$
j = 1		1			-1			1	$\cdots$
j = 2	1		1	-1		-1	1		$\cdots$

Quantum Grothendieck rings (2)

The quantum torus $\mathcal{Y}_t$ associated with $C(z)$ is defined as the $\mathbb{Z}[t^{\pm 1/2}]$-algebra given by

- generators : $\widetilde{Y}_{i,r}^{\pm 1}$ $(i \in I, r \in \mathbb{Z})$
- relations :
  (1) $\widetilde{Y}_{i,r}\widetilde{Y}_{i,r}^{-1} = 1 = \widetilde{Y}_{i,r}^{-1}\widetilde{Y}_{i,r}$,
  (2) for $i, j \in I$ and $r, s \in \mathbb{Z}$,

$$\widetilde{Y}_{i,r}\widetilde{Y}_{j,s} = t^{\gamma(i,r;j,s)}\widetilde{Y}_{j,s}\widetilde{Y}_{i,r},$$

  where $\gamma\colon (I \times \mathbb{Z})^2 \to \mathbb{Z}$ is given by

$$\begin{aligned}\gamma(i,r;j,s) =& \widetilde{c}_{ji}(-r_j - r + s) + \widetilde{c}_{ji}(r_j + r - s) \\ &- \widetilde{c}_{ji}(r_j - r + s) - \widetilde{c}_{ji}(-r_j + r - s).\end{aligned}$$

Quantum Grothendieck rings (2)

The quantum torus $\mathcal{Y}_t$ $\underline{t\text{-commutation relation:}}$ $\widetilde{Y}_{i,r}\widetilde{Y}_{j,s} = t^{\gamma(i,r;j,s)}\widetilde{Y}_{j,s}\widetilde{Y}_{i,r}$

$\gamma(i,r;j,s) = \widetilde{c}_{ji}(-r_j - r + s) + \widetilde{c}_{ji}(r_j + r - s) - \widetilde{c}_{ji}(r_j - r + s) - \widetilde{c}_{ji}(-r_j + r - s)$.

There exists a $\mathbb{Z}$-algebra homomorphism $\mathrm{ev}_{t=1} \colon \mathcal{Y}_t \to \mathcal{Y}$ given by

$$t^{1/2} \mapsto 1 \qquad \widetilde{Y}_{i,r} \mapsto Y_{i,r}.$$

This map is called the specialization at $t = 1$.
There exists a $\mathbb{Z}$-algebra anti-involution $\overline{(\cdot)}$ on $\mathcal{Y}_t$ given by

$$t^{1/2} \mapsto t^{-1/2} \qquad \widetilde{Y}_{i,r} \mapsto t^{-1}\widetilde{Y}_{i,r}.$$

This map is called the bar-involution.

$\forall\ m \in \mathcal{Y}$ monomial $\rightsquigarrow \exists!\ \underline{m} \in \mathcal{Y}_t$ monomial (with coefficient in $t^{\mathbb{Z}}$) such that $\overline{\underline{m}} = \underline{m}$. (e.g. $\underline{Y_{i,r}} = t^{-1/2}\widetilde{Y}_{i,r}$.) Set $\widetilde{A}_{i,r} := \underline{A_{i,r}}$.

Quantum Grothendieck rings (3)

For $i \in I$, set

$$K_{i,t} := \langle \widetilde{Y}_{i,r}(1 + t\widetilde{A}^{-1}_{i,r+r_i}), \widetilde{Y}^{\pm 1}_{j,r} \mid j \in I \setminus \{i\}, r \in \mathbb{Z} \rangle_{\mathbb{Z}[t^{\pm 1/2}]-\mathrm{alg.}} \subset \mathcal{Y}_t.$$

Define the quantum Grothendieck ring of $\mathcal{C}_\bullet$ as

$$K_t(\mathcal{C}_\bullet) := \bigcap_{i \in I} K_{i,t}.$$

Remark

Indeed, $K_{i,t} =$ the kernel of a t-analogue of "the screening operator associated to $i \in I$" [Hernandez]. This is an affine (and t-deformed) analogue of invariance under the simple reflection s_i.

$\rightsquigarrow$ $K_t(\mathcal{C}_\bullet)$ is an affine analogue of the space of "W-invariant functions".

Theorem (Varagnolo-Vasserot, Nakajima, Hernandez)

$\mathrm{ev}_{t=1}(K_t(\mathcal{C}_\bullet)) = K(\mathcal{C}_\bullet)$.

(q,t)-characters (1)

$\exists$ a $\mathbb{Z}[t^{\pm 1/2}]$-basis $\{M_t(m) \mid m \in \mathcal{B}\}$ of $K_t(\mathcal{C}_\bullet)$ such that
$$\mathrm{ev}_{t=1}(M_t(m)) = \chi_q(M(m)) \text{ [Nakajima Hernandez].}$$
$\rightsquigarrow$ $M_t(m)$ is called the (q,t)-character of $M(m)$.

All $M_t(m)$ can be explicitly calculated once we know $M_t(Y_{i,0}), i \in I$.

Theorem (Nakajima (ADE cases), Hernandez (arbitrary))

$\exists! \{L_t(\underline{m}) \mid m \in \mathcal{B}\}$ a $\mathbb{Z}[t^{\pm 1/2}]$-basis of $K_t(\mathcal{C}_\bullet)$ such that

(S1) $\overline{L_t(m)} = L_t(m)$, *and*

(S2) $M_t(m) = L_t(m) + \sum_{m' < m} P_{m,m'}(t) L_t(m')$ *with* $P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$.

The element $L_t(m)$ is called the (q,t)-character of $L(m)$.

(q,t)-characters (2)

(S1) $\overline{L_t(m)} = L_t(m)$ (S2) $M_t(m) = L_t(m) + \sum_{m'<m} P_{m,m'}(t) L_t(m')$, $P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$

Remark

The characterization properties (S1) and (S2) provide an inductive algorithm for computing $P_{m,m'}(t)$'s, called Kazhdan-Lusztig algorithm.

When $\mathfrak{g}$ is of ADE type,

$$\mathrm{ev}_{t=1}(L_t(m)) = \chi_q(L(m)) \text{ [Nakajima].}$$

Its proof is based on his geometric construction using quiver varieties, and it is valid only in ADE case. Moreover, in this case,

$$P_{m,m'}(t) \in t^{-1}\mathbb{Z}_{\geq 0}[t^{-1}] \text{ (positivity).}$$

(q,t)-characters (2)

(S1) $\overline{L_t(m)} = L_t(m)$ (S2) $M_t(m) = L_t(m) + \sum_{m'<m} P_{m,m'}(t) L_t(m')$, $P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$

Remark

The characterization properties (S1) and (S2) provide an inductive algorithm for computing $P_{m,m'}(t)$'s, called Kazhdan-Lusztig algorithm.

Conjecture (Hernandez)

For arbitrary cases, we also have
(1) $\forall m \in \mathcal{B}$, $\mathrm{ev}_{t=1}(L_t(m)) = \chi_q(L(m))$. (2) $P_{m,m'}(t) \in t^{-1}\mathbb{Z}_{\geq 0}[t^{-1}]$.

If Conjecture (1) holds (in particular, in ADE cases), we have

$$[M(m)] = [L(m)] + \sum_{m'<m} P_{m,m'}(1)[L(m')] \text{ in } K(\mathcal{C}_\bullet).$$

T-system

For $i \in I$, $r \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$, set $m^{(i)}_{k,r} := \prod_{s=1}^{k} Y_{i,r+2r_i(s-1)}$. $(m^{(i)}_{1,r} = Y_{i,r})$

$\rightsquigarrow$ the simple module $L(m^{(i)}_{k,r})$ is called a Kirillov-Reshetikhin module.

The T-system of type B [Hernandez]

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}^{(1)}_n$. For $i \in I$, $r \in \mathbb{Z}$, $k \in \mathbb{Z}_{>0}$, the following identity holds in $K(\mathcal{C}_\bullet)$:

$$[L(m^{(i)}_{k,r})][L(m^{(i)}_{k,r+2r_i})] = [L(m^{(i)}_{k+1,r})][L(m^{(i)}_{k-1,r+2r_i})] + [S^{(i)}_{k,r}].$$

Here, $$[S^{(i)}_{k,r}] = \begin{cases} [L(m^{(i-1)}_{k,r+2})][L(m^{(i+1)}_{k,r+2})] \text{ if } i \leq n-2, \\ [L(m^{(n-2)}_{k,r+2})][L(m^{(n)}_{2k,r+1})] \text{ if } i = n-1, \\ [L(m^{(n-1)}_{s,r+1})][L(m^{(n-1)}_{s,r+3})] \text{ if } i = n \text{ and } k = 2s \text{ is even}, \\ [L(m^{(n-1)}_{s+1,r+1})][L(m^{(n-1)}_{s,r+3})] \text{ if } i = n \text{ and } k = 2s+1 \text{ is odd}. \end{cases}$$

$([L(m^{(0)}_{*,*})] := 1)$.

T-system

For $i \in I$, $r \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$, set $m^{(i)}_{k,r} := \prod_{s=1}^{k} Y_{i,r+2r_i(s-1)}$. ($m^{(i)}_{1,r} = Y_{i,r}$)
$\rightsquigarrow$ the simple module $L(m^{(i)}_{k,r})$ is called a Kirillov-Reshetikhin module.

The quantum T-system of type B [Hernandez-O.]

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}^{(1)}_n$. Then $\exists \alpha, \beta \in \mathbb{Z}$ such that the following identity holds in $K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}^{(1)}_n})$ ($\leftarrow$ later) :

$$L_t(m^{(i)}_{k,r})L_t(m^{(i)}_{k,r+2r_i}) = t^{\alpha/2}L_t(m^{(i)}_{k+1,r})L_t(m^{(i)}_{k-1,r+2r_i}) + t^{\beta/2}S^{(i)}_{k,r,t}.$$

Here, $S^{(i)}_{k,r,t} = \begin{cases} L_t(m^{(i-1)}_{k,r+2})L_t(m^{(i+1)}_{k,r+2}) & \text{if } i \leq n-2, \\ L_t(m^{(n-2)}_{k,r+2})L_t(m^{(n)}_{2k,r+1}) & \text{if } i = n-1, \\ L_t(m^{(n-1)}_{s,r+1})L_t(m^{(n-1)}_{s,r+3}) & \text{if } i = n \text{ and } k = 2s \text{ is even,} \\ L_t(m^{(n-1)}_{s+1,r+1})L_t(m^{(n-1)}_{s,r+3}) & \text{if } i = n \text{ and } k = 2s+1 \text{ is odd.} \end{cases}$

$(L_t(m^{(0)}_{*,*}) := 1)$.

Remarks and examples

Remark

- Under the non-quantum settings, the T-systems (similar to the above equalities) have been established also for arbitrary cases [Nakajima, Hernandez].
- Under the quantum settings, the T-systems have been established also for ADE cases [Nakajima, Hernandez-Leclerc].
- In our proof, we use the property

  "thinness of Kirillov-Reshetikhin modules".

  This is true only for type $\mathrm{A}_n^{(1)}$ and $\mathrm{B}_n^{(1)}$.

Example ($\mathrm{B}_3^{(1)}$-case)

- $L_t(m_{2,r}^{(1)})L_t(m_{2,r+4}^{(1)}) = tL_t(m_{3,r}^{(1)})L_t(m_{1,r+4}^{(1)}) + L_t(m_{2,r+2}^{(2)})$.
- $L_t(m_{3,r}^{(3)})L_t(m_{3,r+2}^{(3)}) = t^{1/2}L_t(m_{4,r}^{(3)})L_t(m_{2,r+2}^{(3)}) + t^{-1/2}L_t(m_{2,r+1}^{(2)})L_t(m_{1,r+3}^{(2)})$.

Quantized coordinate algebra of type A_N

Let $\mathcal{U}_v^-$ be the negative half of the QEA of type A_N over $\mathbb{Q}(v^{1/2})$.

$\left(:= \text{the } \mathbb{Q}(v^{1/2})\text{-algebra with } \underline{\text{generators}} \ \{f_i\}_{i=1,\dots,N}, \ \underline{\text{relations}} \begin{cases} f_i^2 f_j - (v+v^{-1}) f_i f_j f_i + f_j f_i^2 = 0 & \text{if } |i-j| = 1 \\ f_i f_j - f_j f_i = 0 & \text{if } |i-j| > 1. \end{cases} \right)$

$\rightsquigarrow \mathcal{A}_v[N_-^{\mathrm{A}_N}] \underset{\mathbb{Z}[v^{\pm 1/2}]\text{-subalg}}{\subset} \mathcal{U}_v^-$ the quantized coordinate algebra.

Property

$\mathbb{Q}(v^{\pm 1/2}) \otimes_{\mathbb{Z}[v^{\pm 1/2}]} \mathcal{A}_v[N_-^{\mathrm{A}_N}] \simeq \mathcal{U}_v^- \quad \mathbb{C} \otimes_{\mathbb{Z}[v^{\pm 1/2}]} \mathcal{A}_v[N_-^{\mathrm{A}_N}] \simeq \mathbb{C}[N_-^{\mathrm{A}_N}]$.

Here $N_-^{\mathrm{A}_N} := \{(N+1) \times (N+1)$ unipotent lower triangular matrices$\}$.

- $\exists \mathrm{ev}_{v=1} \colon \mathcal{A}_v[N_-^{\mathrm{A}_N}] \to \mathbb{C}[N_-^{\mathrm{A}_N}]$ a $\mathbb{Z}$-algebra homomorphism, called the specialization at $v = 1$.
- $\exists$ an $\mathbb{Z}$-algebra anti-involution σ' on $\mathcal{A}_v[N_-^{\mathrm{A}_N}]$, called the (twisted) dual bar involution (e.g. $v^{1/2} \mapsto v^{-1/2}$).

(:= the restriction of the $\mathbb{Z}$-algebra anti-involution on $\mathcal{U}_v^-$ given by $v^{1/2} \mapsto v^{-1/2}$, $f_i \mapsto -f_i$.)

Dual canonical bases

Let $\boldsymbol{i} = (i_1, i_2, \ldots, i_\ell)$ be a reduced word of the longest element w_0 of the Weyl group $W^{\mathrm{A}_N} \simeq \mathfrak{S}_{N+1}$.
(e.g. if $N = 2$, then $\boldsymbol{i} = (1, 2, 1)$ or $(2, 1, 2)$.)

Dual canonical bases

Let $\boldsymbol{i} = (i_1, i_2, \ldots, i_\ell)$ be a reduced word of the longest element w_0 of the Weyl group $W^{\mathrm{A}_N} \simeq \mathfrak{S}_{N+1}$. Let Δ_+ be the set of positive roots of type A_N.
$\rightsquigarrow \exists\{\widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}) \mid \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+}\}$ a $\mathbb{Z}[v^{\pm 1/2}]$-basis of $\mathcal{A}_v[N_-^{\mathrm{A}_N}]$ depending on $\boldsymbol{i}$, which is an analogue of the (dual) PBW-basis associated to $\boldsymbol{i}$ [Lusztig].

Theorem (Lusztig, Saito, Kimura)

- $\exists! \widetilde{\mathbf{B}}^{\mathrm{up}} := \{\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}) \mid \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+}\}$ *a* $\mathbb{Z}[v^{\pm 1/2}]$*-basis of* $\mathcal{A}_v[N_-^{\mathrm{A}_N}]$ *such that*
  - (B1) $\sigma'(\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i})) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i})$, *and*
  - (B2) $\widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}) + \sum_{\boldsymbol{c}'} p_{\boldsymbol{c},\boldsymbol{c}'}(v)\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}', \boldsymbol{i})$ *with* $p_{\boldsymbol{c},\boldsymbol{c}'}(v) \in v\mathbb{Z}[v]$.
- $\widetilde{\mathbf{B}}^{\mathrm{up}}$ *does not depend on the choice of* $\boldsymbol{i}$.

The basis $\widetilde{\mathbf{B}}^{\mathrm{up}}$ is called the (normalized) dual canonical basis.

Positivities

(B1) $\sigma'(\widetilde{G^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i})) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i})$ (B2) $\widetilde{F^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i}) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i}) + \sum_{\boldsymbol{c}'} p_{\boldsymbol{c},\boldsymbol{c}'}(v)\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}',\boldsymbol{i}), p_{\boldsymbol{c},\boldsymbol{c}'}(v) \in v\mathbb{Z}[v]$

Theorem (Lusztig ($\boldsymbol{i}$ "adapted"), Kato, McNamara (arbitrary), (O. arbitrary))

$p_{\boldsymbol{c},\boldsymbol{c}'}(v) \in \mathbb{Z}_{\geq 0}[v]$.

Theorem (Lusztig)

For $\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathbb{Z}_{\geq 0}^{\Delta_+}$, *write*

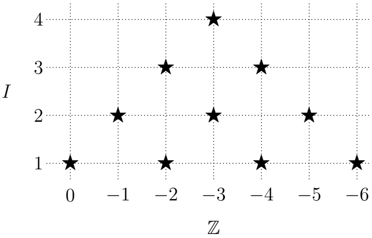
$$\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}_1,\boldsymbol{i})\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}_2,\boldsymbol{i}) = \sum_{\boldsymbol{c}} c^{\boldsymbol{c}}_{\boldsymbol{c}_1,\boldsymbol{c}_2}\widetilde{G^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i}).$$

Then $c^{\boldsymbol{c}}_{\boldsymbol{c}_1,\boldsymbol{c}_2} \in \mathbb{Z}_{\geq 0}[v^{\pm 1/2}]$.

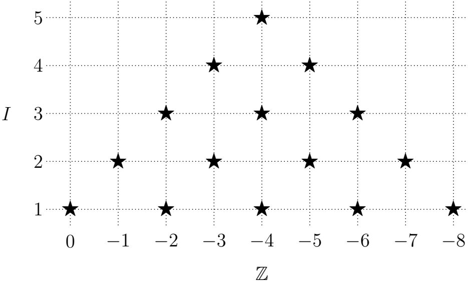
Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{A}_N^{(1)}$ ($I = \{1, \dots, N\}$).
Define $J_{\mathcal{Q}', \mathrm{A}_N^{(1)}}$ by

$$J_{\mathcal{Q}', \mathrm{A}_N^{(1)}} := \{(\imath, -\imath+1-2k) \in I \times \mathbb{Z} \mid k = 0, 1, \dots, 2n-\imath-1 \text{ and } \imath \in I\}.$$

$\underline{N = 4}$



$\underline{N = 5}$



Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{A}_N^{(1)}$ ($I = \{1, \dots, N\}$).
Define $J_{\mathcal{Q}', \mathrm{A}_N^{(1)}}$ by

$$J_{\mathcal{Q}', \mathrm{A}_N^{(1)}} := \{(\imath, -\imath+1-2k) \in I \times \mathbb{Z} \mid k = 0, 1, \dots, 2n-\imath-1 \text{ and } \imath \in I\}.$$

Set

$$\mathcal{B}_{\mathcal{Q}', \mathrm{A}_N^{(1)}} := \left\{ \prod_{(\imath, r)} Y_{\imath, r}^{u_{\imath, r}} \in \mathcal{B} \mid u_{\imath, r} \neq 0 \text{ only if } (\imath, r) \in J_{\mathcal{Q}', \mathrm{A}_N^{(1)}} \right\},$$

$\mathcal{C}_{\mathcal{Q}', \mathrm{A}_N^{(1)}}$:=the full subcategory of $\mathcal{C}_\bullet$ such that

$$\underline{\text{object}} : V \text{ with } [V] \in \sum_{m \in \mathcal{B}_{\mathcal{Q}', \mathrm{A}_N^{(1)}}} \mathbb{Z}[L(m)].$$

Lemma (Hernandez-Leclerc)

$\mathcal{C}_{\mathcal{Q}', \mathrm{A}_N^{(1)}}$ is an abelian $\otimes$-subcategory.

Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (2)

Remark

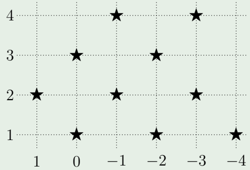
Here we did not mention the meaning of $\mathcal{Q}'$. In fact, here $\mathcal{Q}'$ is the following Dynkin quiver of type A_N

$$1 \longleftarrow 2 \longleftarrow \cdots\cdots \longleftarrow N-1 \longleftarrow N$$

The "arrangement" of $J_{\mathcal{Q}',\mathrm{A}_N^{(1)}}$ is arising from the Auslander-Reiten quiver of $\mathbb{C}\mathcal{Q}'$-mod.

Actually, for any Dynkin quiver $\mathcal{Q}'$ of type $\mathrm{A}_N^{(1)}$, the abelian $\otimes$-subcategory $\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}$ is defined, and all results in the following hold.

<u>e.g.</u> If $\mathcal{Q}'$ is $1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4$, then $J_{\mathcal{Q}',\mathrm{A}_N^{(1)}}$ is described as follows :



Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (2)

Remark

Here we did not mention the meaning of $\mathcal{Q}'$. In fact, here $\mathcal{Q}'$ is the following Dynkin quiver of type A_N

$$1 \longleftarrow 2 \longleftarrow - - - - - - - - - - \longleftarrow N-1 \longleftarrow N$$

The "arrangement" of $J_{\mathcal{Q}',\mathrm{A}_N^{(1)}}$ is arising from the Auslander-Reiten quiver of $\mathbb{C}\mathcal{Q}'$-mod.

Actually, for any Dynkin quiver $\mathcal{Q}'$ of type $\mathrm{A}_N^{(1)}$, the abelian $\otimes$-subcategory $\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}$ is defined, and all results in the following hold.

The number of variants of such subcategories (up to shift of the spectral parameter) is 2^{N-1}.

Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (3)

Set

$$K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

Lemma

$K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}})$ is a $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of $K_t(\mathcal{C}_\bullet)$.

$\rightsquigarrow$ $K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}})$ is called the quantum Grothendieck ring of $\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}$.

Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (3)

Set

$$K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}) := \sum_{m\in\mathcal{B}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}]M_t(m) = \sum_{m\in\mathcal{B}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}]L_t(m).$$

Write

$J_{\mathcal{Q}',\mathrm{A}_N^{(1)}} = \{(\imath_s, r_s) \mid s = 1,\dots,\ell(= N(N+1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathcal{Q}'} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_N}$.

Remark

The reduced word $\boldsymbol{i}_{\mathcal{Q}'}$ depends on the choice of the total ordering on $J_{\mathcal{Q}',\mathrm{A}_N^{(1)}}$. However, its "commutation class" is uniquely determined.
The following results does not depend on this choice.
This $\boldsymbol{i}_{\mathcal{Q}'}$ is "adapted to $\mathcal{Q}'$".

Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (3)

Set

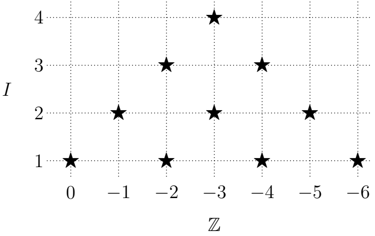
$$K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

$J_{\mathcal{Q}',\mathrm{A}_N^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= N(N+1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathcal{Q}'} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_N}$.

In the following example, $\boldsymbol{i}_{\mathcal{Q}'} = (1, 2, 1, 3, 2, 4, 1, 3, 2, 1)$ etc.

$\underline{N = 4}$



Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (3)

$$K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q}',\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

$J_{\mathcal{Q}',\mathrm{A}_N^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= N(N+1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathcal{Q}'} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_N}$.

Theorem (Hernandez-Leclerc)

There exists a $\mathbb{Z}$-algebra isomorphism

$$\Phi_{\mathrm{A}} \colon \mathcal{A}_v[N_-^{\mathrm{A}_N}] \xrightarrow{\sim} K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_N^{(1)}})$$

given by

$$v^{\pm 1/2} \mapsto t^{\mp 1/2} \qquad \widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathcal{Q}'}) \mapsto M_t(m(\boldsymbol{c})) \ \forall \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+},$$

here $m(\boldsymbol{c}) = \prod_{k=1}^{\ell} Y_{\imath_k, r_k}^{\boldsymbol{c}(s_{\imath_1} \cdots s_{\imath_{k-1}} \alpha_{\imath_k})}$. *Moreover,*

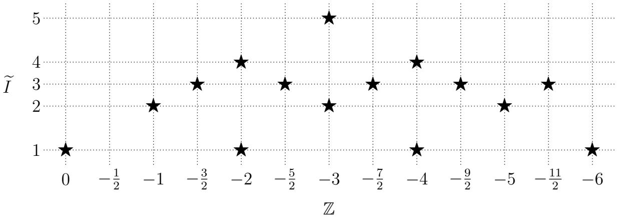
$$\Phi_{\mathrm{A}}(\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathcal{Q}'})) = L_t(m(\boldsymbol{c})). \ \forall \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+}.$$

Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}_n^{(1)}$ $(I = \{1, \ldots, n\})$.
Let $\widetilde{I} := \{1, \ldots, 2n-1\}$. Define $\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ by

$$\begin{aligned}\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}} :=& \{(\imath, -\imath + 2 - 2k) \mid k = 0, \ldots, 2n-1-\imath \text{ and } \imath = n+1, \ldots, 2n-1\} \\ &\cup \{(n, -n + \frac{3}{2} - k) \mid k = 0, \ldots, 2n-2\} \\ &\cup \{(\imath, -\imath + 1 - 2k) \mid k = 0, \ldots, 2n-2-\imath \text{ and } \imath = 1, \ldots, n-1\}.\end{aligned}$$

$\underline{n = 3}$

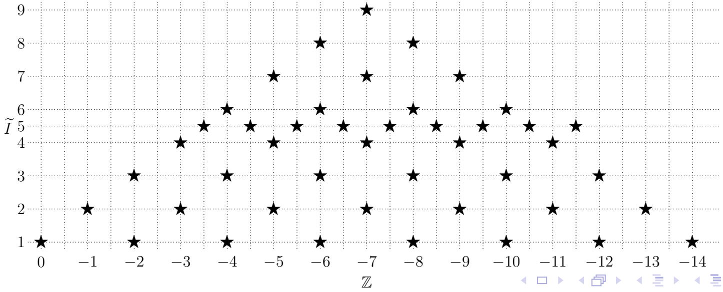


Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}_n^{(1)}$ ($I = \{1, \dots, n\}$).
Let $\widetilde{I} := \{1, \dots, 2n-1\}$. Define $\widetilde{J}_{\mathcal{Q}, \mathrm{B}_n^{(1)}}$ by

$$
\begin{aligned}
\widetilde{J}_{\mathcal{Q}, \mathrm{B}_n^{(1)}} := & \{(\imath, -\imath + 2 - 2k) \mid k = 0, \dots, 2n-1-\imath \text{ and } \imath = n+1, \dots, 2n-1\} \\
& \cup \{(n, -n + \frac{3}{2} - k) \mid k = 0, \dots, 2n-2\} \\
& \cup \{(\imath, -\imath + 1 - 2k) \mid k = 0, \dots, 2n-2-\imath \text{ and } \imath = 1, \dots, n-1\}.
\end{aligned}
$$

$\underline{n = 5}$



Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}_n^{(1)}$ ($I = \{1, \dots, n\}$).
Let $\widetilde{I} := \{1, \dots, 2n-1\}$. Define $\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$.

Consider the map $\tilde{I} \to I, \imath \mapsto \bar{\imath} := \begin{cases} \imath & \text{if } \imath \leq n, \\ 2n - \imath & \text{if } \imath > n. \end{cases}$ "folding"

Set

$$\mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}} := \left\{ \prod_{(i,r)} Y_{i,r}^{u_{i,r}} \in \mathcal{B} \;\middle|\; \begin{array}{l} u_{i,r} \neq 0 \text{ only if } (i,r) = (\bar{\imath}, 2s) \\ \text{for some } (\imath, s) \in \widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}} \end{array} \right\},$$

$\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$:=the full subcategory of $\mathcal{C}_\bullet$ such that

$$\underline{\text{object}} : V \text{ with } [V] \in \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[L(m)].$$

Lemma (Oh-Suh, Hernandez-O.)

$\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ *is an abelian* $\otimes$*-subcategory.*

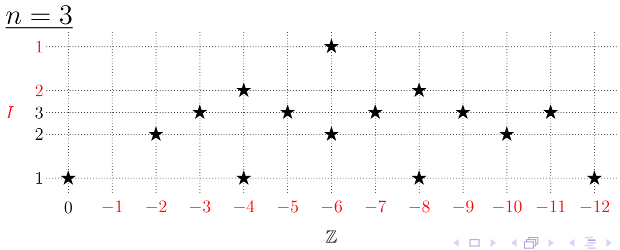
Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}_n^{(1)}$ $(I = \{1, \dots, n\})$.
Let $\widetilde{I} := \{1, \dots, 2n-1\}$.

Consider the map $\tilde{I} \to I, \imath \mapsto \bar{\imath} := \begin{cases} \imath & \text{if } \imath \leq n, \\ 2n - \imath & \text{if } \imath > n. \end{cases}$ "folding"

Set

$$\mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}} := \left\{ \prod\nolimits_{(i,r)} Y_{i,r}^{u_{i,r}} \in \mathcal{B} \;\middle|\; \begin{array}{l} u_{i,r} \neq 0 \text{ only if } (i,r) = (\bar{\imath}, 2s) \\ \text{for some } (\imath, s) \in \widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}} \end{array} \right\}.$$



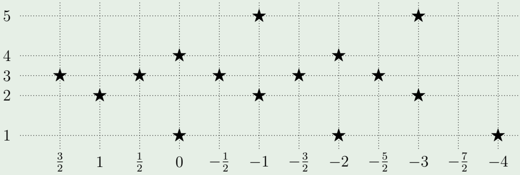
Our isomorphisms (2)

Remark

Here we did not mention the meaning of $\mathcal{Q}$. In fact, $\mathcal{Q} = (\mathcal{Q}', <)$, here $\mathcal{Q}'$ is the previous Dynkin quiver of type A_{2n-2} and $<$ is the "auxiliary datum".

In fact, if we remove the points on "n-th row" from $\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$, then we obtain $J_{\mathcal{Q}',\mathrm{A}_{2n-2}^{(1)}}$. The datum $<$ indicates the place of "extremal points" on n-th row. This correspondence can be generalized to arbitrary $\mathcal{Q}'$ [Oh-Suh].

<u>e.g.</u> If $\mathcal{Q}'$ is $1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4$, then $\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ with $\mathcal{Q} = (\mathcal{Q}', >)$ is described as follows :



Our isomorphisms (2)

Remark

Here we did not mention the meaning of $\mathcal{Q}$. In fact, $\mathcal{Q} = (\mathcal{Q}', <)$, here $\mathcal{Q}'$ is the previous Dynkin quiver of type A_{2n-2} and $<$ is the "auxiliary datum".

In fact, if we remove the points on "n-th row" from $\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$, then we obtain $J_{\mathcal{Q}',\mathrm{A}_{2n-2}^{(1)}}$. The datum $<$ indicates the place of "extremal points" on n-th row. This correspondence can be generalized to arbitrary $\mathcal{Q}'$ [Oh-Suh].

The number of variants of $\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ (up to shift) is $2^{2n-3} \times 2 = 2^{2n-2}$, and the corresponding nice subcategory $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ is always defined.

Our isomorphisms (3)

Set

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

Lemma

$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}})$ is a $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of $K_t(\mathcal{C}_\bullet)$.

$\rightsquigarrow$ $K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}})$ is called the quantum Grothendieck ring of $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$.

Our isomorphisms (3)

Set

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

Write

$\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \ldots, \ell (= 2n(2n-1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}} := (\imath_1, \imath_2, \ldots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_{2n-1}}$.

Remark

The reduced word $\boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}}$ depends on the choice of the total ordering on $J_{\mathcal{Q},\mathrm{B}_n^{(1)}}$. However, its "commutation class" is uniquely determined.
The following results does not depend on this choice.
This $\boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}}$ is always "non-adapted".

Our isomorphisms (3)

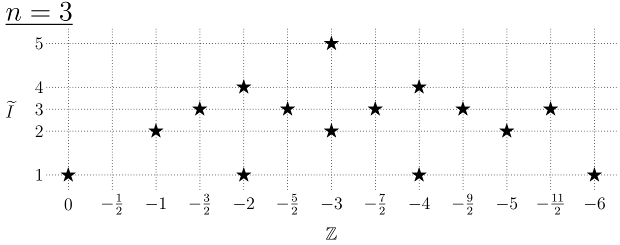
Set

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

$\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= 2n(2n-1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_{2n-1}}$.

In the following example, $\boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}} = (1,2,3,1,4,3,2,5,3,1,4,3,2,3,1)$ etc.



Our isomorphisms (3)

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

$\widetilde{J}_{\mathcal{Q},\mathrm{B}_n^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= 2n(2n-1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_{2n-1}}$.

Theorem (Hernandez-O.)

There exists a $\mathbb{Z}$-algebra isomorphism

$$\Phi_{\mathrm{B}} \colon \mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] \xrightarrow{\sim} K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}})$$

given by

$$v^{\pm 1/2} \mapsto t^{\mp 1/2} \qquad \widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}}) \mapsto M_t(m'(\boldsymbol{c}))\ \forall \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+},$$

here $m'(\boldsymbol{c}) = \prod_{k=1}^{\ell} Y_{\imath_k, r_k}^{\boldsymbol{c}(s_{\imath_1} \cdots s_{\imath_{k-1}} \alpha_{\imath_k})}$. *Moreover,*

$$\Phi_{\mathrm{B}}(\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}})) = L_t(m'(\boldsymbol{c})).\ \forall \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+}.$$

Positivities in $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$

By our theorem, the positivities of the dual canonical bases $\widetilde{\mathbf{B}}^{\mathrm{up}}$ can be transported to those of (q,t)-characters.

Corollary (Positivity of Kazhdan-Lusztig type polynomials)

For $m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$, *write*

$$M_t(m) = \sum_{m' \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} P_{m,m'}(t) L_t(m').$$

as before. Then $P_{m,m'}(t) \in \mathbb{Z}_{\geq 0}[t^{-1}]$.

This is the affirmative answer to Conjecture (2) for $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$.

Corollary (Positivity of structure constants)

For $m_1, m_2 \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$, *write*

$$L_t(m_1)L_t(m_2) = \sum_{\in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}} c^m_{m_1,m_2} L_t(m).$$

Then we have $c^m_{m_1,m_2} \in \mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$.

Comparison with Kashiwara-Oh

The following remarkable theorem is recently proved by means of the celebrated *generalized quantum affine Schur-Weyl dualities*, which is developed by Kang, Kashiwara, Kim and Oh :

Theorem (Kashiwara-Oh '17)

There exists a $\mathbb{Z}$*-algebra isomorphism*

$$[\mathscr{F}] \colon \mathrm{ev}_{v=1}(\mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}]) \xrightarrow{\sim} K(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}})$$

which maps the dual canonical basis $\mathrm{ev}_{v=1}(\widetilde{\mathbf{B}}^{\mathrm{up}})$ *specialized at* $v=1$ *to the set of classes of simple modules* $\{[L(m)] \mid m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}\}$.

Theorem (Hernandez-O.)

$$\Phi_{\mathrm{B}}\,|_{v=t=1} = [\mathscr{F}].$$

Comparison with Kashiwara-Oh

Theorem (Hernandez-O.)

$$\Phi_{\mathrm{B}} \,|_{v=t=1} = [\mathscr{F}].$$

Remark

Our construction of Φ_{B} does not imply Kashiwara-Oh's theorem because, a priori,

- $\Phi_{\mathrm{B}}|_{v=t=1}$ maps $\mathrm{ev}_{v=1}(\widetilde{\mathbf{B}}^{\mathrm{up}})$ to $\{\mathrm{ev}_{v=1}(L_t(m)) | m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}\}$, but
- $[\mathscr{F}]$ maps $\mathrm{ev}_{v=1}(\widetilde{\mathbf{B}}^{\mathrm{up}})$ to $\{[L(m)] \mid m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}\}$,

(The coincidence of these images is nothing but Hernandez's conjecture (1) !) Hence our result and Kashiwara-Oh's result are independent.

Our comparison theorem above is proved by looking at the images of dual PBW-bases.

Comparison with Kashiwara-Oh

Theorem (Kashiwara-Oh '17)

There exists a $\mathbb{Z}$-algebra isomorphism

$$[\mathscr{F}]\colon \mathrm{ev}_{v=1}(\mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}]) \xrightarrow{\sim} K(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}})$$

which maps the dual canonical basis $\mathrm{ev}_{v=1}(\widetilde{\mathbf{B}}^{\mathrm{up}})$ specialized at $v=1$ to the set of classes of simple modules $\{[L(m)] \mid m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}\}$.

Theorem (Hernandez-O.)

$$\Phi_{\mathrm{B}}\mid_{v=t=1} = [\mathscr{F}].$$

Corollary

$\chi_q(L(m)) = \mathrm{ev}_{t=1}(L_t(m)), \forall m \in \mathcal{B}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$.

This is the affirmative answer to Conjecture (1) for $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$.

Comments on further results and proofs (1)

By combining our Φ_{B} with Φ_{A} for $\mathrm{A}_{2n-1}^{(1)}$, we can obtain a $\mathbb{Z}[v^{\pm 1/2}]$-algebra isomorphism $K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{A}_{2n-1}^{(1)}}) \simeq K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}})$. This isomorphism preserves the set of (q,t)-characters of simple modules. (It does not preserve the set of (q,t)-characters of standard modules.) Moreover, if we go to $\mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}]$ via Φ_{A} and Φ_{B}, then

"highest monomial parametrization of $L_t(m)$'s" $\mapsto$
"PBW-parametrization of $\widetilde{\mathbf{B}}^{\mathrm{up}}$"

Hence, the correspondence between simple modules above $=$ the change of PBW-parametrizations. For the choices of $\mathcal{C}_{\mathcal{Q}',\mathrm{A}_{2n-1}^{(1)}}$ and $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ in this talk, we know explicit braid moves between $\boldsymbol{i}_{\mathcal{Q}'}$ and $\boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}}$.
(e.g. $\underline{\mathrm{A}_3^{(1)}/\mathrm{B}_2^{(1)}}$ $\boldsymbol{i}_{\mathcal{Q}'} = (1,2,3,1,2,1) \leftrightarrow \boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}} = (1,2,3,2,1,2)$.)
$\rightsquigarrow$ We can describe the explicit correspondence of simple modules !

Comments on further results and proofs (2)

Sketch of the proof of the existence of Φ_{B}

0) We have

- $K_t(\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}) \overset{\text{“truncate”}}{\hookrightarrow}$ the quantum torus of finitely many variables.
- $\mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] \hookrightarrow$ the quantum torus arising from the “quantum initial seed” associated with $\boldsymbol{i}_{\mathcal{Q}}^{\mathrm{tw}}$ ($\Leftarrow$ quantum cluster algebra).

1) Prove the isomorphism between ambient tori in Step 0. (Here we also use the cluster algebraic observation “$A_{i,r}$'s are $\hat{Y}$-variables”)

2) Show the coincidence between quantum T-system and quantum determinantal ientities ($\Leftarrow$ mutation sequence. Every algebra generator appears as a cluster variable in this sequence).

Reference : arXiv:1803.06754v1