# Quantum Grothendieck ring isomorphisms for quantum affine algebras of type A and B

Hironori OYA (大矢 浩徳)

Shibaura Institute of Technology

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# Motivation (1)

Topic : Finite dimensional representations of affine quantum groups

**Question 1**

*Dimensions/$q$-characters of simple modules ?*

- $\exists$ Classification of simple modules [Chari-Pressley 1990's]
  "Highest weight theory"
- However, there are NO known closed formulae of their dimensions and $q$-characters in general. (e.g. $\nexists$ analogue of Weyl-Kac character formulae...)

**Question 2**

*Description of Grothendieck rings and their "deformations" ?*

- Some (deformed) Grothendieck rings are known to be described nicely as (quantum) cluster algebras...

# Motivation (2)

## Question 1

*Dimensions/$q$-characters of simple modules ?*

- ADE case $\exists$ algorithm to compute them ! [Nakajima '04]
  "Kazhdan-Lusztig algorithm"
  The tool is $t$-deformed $q$-character, and the geometric construction (via graded quiver varieties) of simple modules guarantees this algorithm.
- Arbitrary (untwisted) case [Hernandez '04]
  - $\exists$ $t$-deformed $q$-characters, defined algebraically ($\nexists$ geometry for non-symmetric cases)
  - Kazhdan-Lusztig algorithm gives conjectural $q$-characters of simple modules

However, they are still candidates in non-symmetric cases.

# Motivation (3)

## Question 2

*Description of Grothendieck rings and their "deformations" ?*

Hernandez-Leclerc's following result is a prototype of our work :
Assume that $\underline{\mathrm{X} = \mathrm{ADE}}$. Let

- $K_t(\mathcal{C}_{\mathcal{Q},\mathrm{X}_n^{(1)}})$ the $t$-deformed Grothendieck ring (=quantum Grothendieck ring) associated to an abelian monoidal category $\mathcal{C}_{\mathcal{Q},\mathrm{X}_n^{(1)}}$ for type $\mathrm{X}_n^{(1)}$
- $\mathcal{A}_v[N_-^{\mathrm{X}_n}]$ the quantized coordinate algebra of the unipotent group of type $\mathrm{X}_n$

## Theorem (Hernandez-Leclerc '15)

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{X}_n^{(1)}}) \simeq \mathcal{A}_v[N_-^{\mathrm{X}_n}], \left\{ \begin{array}{c} (q,t)\textit{-characters of} \\ \textit{simple modules} \end{array} \right\} \leftrightarrow \textit{dual canonical basis.}$$

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## Question 2

*Description of Grothendieck rings and their "deformations" ?*

If $\underline{\mathrm{X} = \mathrm{ADE}}$, then $K_t(\mathcal{C}_{\mathcal{Q},\mathrm{X}_n^{(1)}}) \overset{\text{[HL]}}{\simeq} \mathcal{A}_v[N_-^{\mathrm{X}_n}]$, $\left\{ \begin{array}{c} (q,t)\text{-characters of} \\ \text{simple modules} \end{array} \right\} \leftrightarrow$ dual canonical basis.

Why is it a "nice" description ?

- $\mathcal{A}_v[N_-^{\mathrm{X}_n}]$ can be described by generators and relations (after an appropriate extension of coefficients).
- $\mathcal{A}_v[N_-^{\mathrm{X}_n}]$ is known to be a "quantum cluster algebra" [Geiß-Leclerc-Schröer], and their quantum cluster monomials are known to be contained in the dual canonical basis [Kang-Kashiwara-Kim-Oh, Qin].

$\exists$ an analogue of this isomorphism in the non-symmetric case ?

# Overview of Main results

In this talk, we consider the case of type $\mathrm{B}_n^{(1)}$. Let $\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$ be the monoidal subcategory $\mathcal{C}_{\mathcal{Q},\mathrm{X}_n^{(1)}}$ for type $\mathrm{B}_n^{(1)}$.

## Theorem (Hernandez-O.)

$$\begin{array}{ccccc}
K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}) & \simeq & \mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] & \overset{[\mathrm{HL}]}{\simeq} & K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_{2n-1}^{(1)}}) \\
\cup & & \cup & & \cup \\
\left\{ \begin{array}{c} (q,t)\text{-}\textit{characters of} \\ \textit{simple modules} \end{array} \right\} & \leftrightarrow & \textit{dual canonical basis} & \overset{[\mathrm{HL}]}{\longleftrightarrow} & \left\{ \begin{array}{c} (q,t)\text{-}\textit{characters of} \\ \textit{simple modules} \end{array} \right\}
\end{array}$$

## Remark

There are no known direct algebraic relations between the quantum affine algebras of type $\mathrm{B}_n^{(1)}$ and $\mathrm{A}_{2n-1}^{(1)}$ themselves.

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Kashiwara-Oh established an isomorphism between $K_{t=1}(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}})$ and $\mathbb{C}[N_-^{\mathrm{A}_{2n-1}}]$ by a different method. Combining this result with our theorem above, we obtain the following :

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\end{array}$$

**Theorem (Hernandez-O.)**

*The $(q,t)$-characters of simple modules in $\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$ specialize to the corresponding $q$-characters.*

$\rightsquigarrow$ The Kazhdan-Lusztig algorithm gives "correct" answers in $\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$!

# Quantum affine algebras

Let

- $\mathfrak{g}$ a finite dimensional simple Lie algebra / $\mathbb{C}$
- $\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ its loop algebra $[X \otimes t^m, Y \otimes t^m] = [X, Y] \otimes t^{m+m'}$
- $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ the Drinfeld-Jimbo quantum loop algebra / $\mathbb{C}$ with a parameter $q \in \mathbb{C}^\times$ not a root of unity
  generators : $\{k_i^{\pm 1}, x_{i,r}^{\pm}, h_{i,s} \mid i \in I, r \in \mathbb{Z}, s \in \mathbb{Z} \setminus \{0\}\}$

$\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ has a Hopf algebra structure.

Let $\mathcal{C}_{\mathrm{X}_n^{(1)}}$ be the category of finite-dimensional $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$-modules of type 1 (i.e. the eigenvalues of the actions of $\{k_i \mid i \in I\}$ are of the form $q^m$, $m \in \mathbb{Z}$).

$\mathcal{C}_{\mathrm{X}_n^{(1)}}$ is a non-semisimple abelian $\otimes$-category.

# $q$-characters (1)

Let $V \in \mathcal{C}_{\mathrm{X}_n^{(1)}}$. Frenkel-Reshetikhin showed that

$$\{\text{Generalized simultaneous eigenvalues of all } k_i^{\pm 1}, h_{i,s} \curvearrowright V\ \} \hookrightarrow \{\text{Laurent monomials } m \text{ in } Y_{i,a}\text{'s } (i \in I, a \in \mathbb{C}^\times)\ \}$$

$\rightsquigarrow V = \bigoplus_m V_m$, called the $\ell$-weight space decomposition.

$Y_{i,a}$ is an "affine analogue" of $e^{\varpi_i}$, $\varpi_i$ fundamental weight.

Define the $q$-character of $V$ as

$$\chi_q(V) := \sum_m \dim(V_m) m.$$

Then

$$\chi_q \colon K(\mathcal{C}_{\mathrm{X}_n^{(1)}}) \overset{\text{alg.}}{\hookrightarrow} \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^\times] =: \mathcal{Y}_{\mathbb{C}^\times},$$

here $K(\mathcal{C}_{\mathrm{X}_n^{(1)}})$ be the Grothendieck ring of $\mathcal{C}_{\mathrm{X}_n^{(1)}}$ [Frenkel-Reshetikhin].

$K(\mathcal{C}_{\mathrm{X}_n^{(1)}})$ is commutative. (However sometimes $V \otimes W \not\simeq W \otimes V$.)

# $q$-characters (2)

Set $\mathcal{B}_{\mathbb{C}^\times} := \left\{ \prod_{i\in I, a\in\mathbb{C}^\times} Y_{i,a}^{m_{i,a}} \mid m_{i,a} \geq 0 \right\} \subset \mathcal{Y}_{\mathbb{C}^\times}$ dominant monomials.

## Classification of simple modules [Chari-Pressley]

There is a one-to-one correspondence :

$$\begin{array}{ccc} \left\{\text{simple modules in } \mathcal{C}_{\mathrm{X}_n^{(1)}}\right\} / \sim & \leftrightarrow & \mathcal{B}_{\mathbb{C}^\times} \\ \Cup & & \Cup \\ [L(m)] & \leftrightarrow & m \end{array}$$

$\exists$ an "affine analogue" $A_{i,a} \in \mathcal{Y}_{\mathbb{C}^\times}$ of $e^{\alpha_i}$, $\alpha_i$ simple root.

Type $\mathrm{A}_n^{(1)}$

$$A_{i,a} = Y_{i,aq^{-1}} Y_{i,aq} Y_{i-1,a}^{-1} Y_{i+1,a}^{-1} \left( \leftrightsquigarrow e^{\alpha_i} = e^{2\varpi_i - \varpi_{i-1} - \varpi_{i+1}} \right)$$

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Type $\mathrm{B}_n^{(1)}$

$$A_{i,a} = \begin{cases} Y_{i,aq^{-2}} Y_{i,aq^2} Y_{i-1,a}^{-1} Y_{i+1,a}^{-1} & \text{if } i \leq n-2 \\ Y_{n-1,aq^{-2}} Y_{n-1,aq^2} Y_{n-2,a}^{-1} Y_{n,aq^{-1}}^{-1} Y_{n,aq}^{-1} & \text{if } i = n-1 \\ Y_{n,aq^{-1}} Y_{n,aq} Y_{n-1,a}^{-1} & \text{if } i = n. \end{cases}$$

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Define the partial ordering on the set of Laurent monomials in $\mathcal{Y}_{\mathbb{C}^\times}$ as

$$m \geq m' \Leftrightarrow m^{-1}m' \text{ is a product of } A_{i,a}^{-1}\text{'s.}$$

## Theorem (Frenkel-Mukhin)

$$\chi_q(L(m)) = m + \textit{(sum of terms lower than } m\textit{)}, \ \forall m \in \mathcal{B}_{\mathbb{C}^\times}.$$

# $q$-characters (3)

$\mathcal{C}_{\bullet,\mathrm{X}_n^{(1)}}$ :=the full subcategory of $\mathcal{C}_{\mathrm{X}_n^{(1)}}$ such that

<u>object</u> : $V$ with $\chi_q(V) \in \mathbb{Z}[Y_{i,q^r}^{\pm 1} \mid i \in I, r \in \mathbb{Z}] =: \mathcal{Y}$.

$$\mathcal{C}_{\mathrm{X}_n^{(1)}} = \bigotimes_{a \in \mathbb{C}^\times / q^{\mathbb{Z}}} \left( \mathcal{C}_{\bullet,\mathrm{X}_n^{(1)}} \right)_a$$ ($\left( \mathcal{C}_{\bullet,\mathrm{X}_n^{(1)}} \right)_a$ is almost a copy of $\mathcal{C}_{\bullet,\mathrm{X}_n^{(1)}}$).

From now on, we always work in $\mathcal{C}_{\bullet,\mathrm{X}_n^{(1)}}$, and write

$$Y_{i,r} := Y_{i,q^r} \qquad A_{i,r} := A_{i,q^r} \qquad \mathcal{B} := \mathcal{B}_{\mathbb{C}^\times} \cap \mathcal{Y}.$$

## Example

- $\mathfrak{g} = \mathfrak{sl}_2, I = \{1\}$, $\chi_q(L(Y_{1,r})) = Y_{1,r} + Y_{1,r+2}^{-1} = Y_{1,r}(1 + A_{1,r+1}^{-1})$.
- $\mathfrak{g} = \mathfrak{so}_5, I = \{1, 2\}$,

$\chi_q(L(Y_{1,r})) = Y_{1,r} + Y_{2,r+1}Y_{2,r+3}Y_{1,r+4}^{-1} + Y_{2,r+1}Y_{2,r+5}^{-1} + Y_{1,r+2}Y_{2,r+3}^{-1}Y_{2,r+5}^{-1} + Y_{1,r+6}^{-1}$.

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For $m = \prod_{i \in I, r \in \mathbb{Z}} Y_{i,r}^{u_{i,r}} \in \mathcal{B}$, a standard module is defined as

$$M(m) := \overrightarrow{\bigotimes_{r \in \mathbb{Z}}} \left( \bigotimes_{i \in I} L(Y_{i,r})^{\otimes u_{i,r}} \right).$$

$\{[L(m)] \mid m \in \mathcal{B}\}$ and $\{[M(m)] \mid m \in \mathcal{B}\}$ are $\mathbb{Z}$-bases of $K(\mathcal{C}_{\bullet,\mathrm{X}_n^{(1)}})$, and the latter is relatively easy. Want : the transition matrices.

# Quantum Grothendieck rings (1)

We follow Hernandez's algebraic construction of quantum Grothendieck rings here.

## Remark

$\exists$ other (geometric) constructions given by Varagnolo-Vasserot and Nakajima for $\mathrm{ADE}$ cases, and all constructions produce equivalent rings in these cases.

First, we prepare a deformation $\mathcal{Y}_t$ of the ambient Laurent polynomial ring $\mathcal{Y}$.

$\rightsquigarrow$ $\mathcal{Y}_t$ is a $\mathbb{Z}[t^{\pm 1/2}]$-algebra such that

- generators : $\widetilde{Y}_{i,r}$ $(i \in I, r \in \mathbb{Z})$ and their inverses $\widetilde{Y}_{i,r}^{-1}$
- relations : $\widetilde{Y}_{i,r}$'s mutually $t$-commute.

e.g. $\mathrm{B}_2^{(1)}$-case : $\widetilde{Y}_{1,r+2}\widetilde{Y}_{1,r} = t\widetilde{Y}_{1,r}\widetilde{Y}_{1,r+2}$, $\widetilde{Y}_{1,r+5}\widetilde{Y}_{2,r} = t^{-1}\widetilde{Y}_{2,r}\widetilde{Y}_{1,r+5}, \dots$

# Quantum Grothendieck rings (2)

There exists a $\mathbb{Z}$-algebra homomorphism $\mathrm{ev}_{t=1}\colon \mathcal{Y}_t \to \mathcal{Y}$ given by

$$t^{1/2} \mapsto 1 \qquad \widetilde{Y}_{i,r} \mapsto Y_{i,r}.$$

This map is called the specialization at $t = 1$.
There exists a $\mathbb{Z}$-algebra anti-involution $\overline{(\cdot)}$ on $\mathcal{Y}_t$ given by

$$t^{1/2} \mapsto t^{-1/2} \qquad \widetilde{Y}_{i,r} \mapsto t^{-1}\widetilde{Y}_{i,r}.$$

This map is called the bar-involution.

$\forall\ m \in \mathcal{Y}$ monomial $\rightsquigarrow \exists!\ \underline{m} \in \mathcal{Y}_t$ monomial (with coefficient in $t^{\mathbb{Z}/2}$) such that $\overline{\underline{m}} = \underline{m}$. (e.g. $\underline{Y_{i,r}} = t^{-1/2}\widetilde{Y}_{i,r}$.) Set $\widetilde{A}_{i,r} := \underline{A_{i,r}}$.

# Quantum Grothendieck rings (3)

For $i \in I$, set

$$K_{i,t} := \langle \widetilde{Y}_{i,r}(1 + t\widetilde{A}_{i,r+r_i}^{-1}), \widetilde{Y}_{j,r}^{\pm 1} \mid j \in I \setminus \{i\}, r \in \mathbb{Z} \rangle_{\mathbb{Z}[t^{\pm 1/2}]-\mathrm{alg.}} \subset \mathcal{Y}_t.$$

Define the quantum Grothendieck ring of $\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}}$ as

$$K_t(\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}}) := \bigcap_{i \in I} K_{i,t}.$$

Note that $\overline{K_t(\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}})} = K_t(\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}})$.

## Remark

Indeed, $K_{i,t} =$ the kernel of a $t$-analogue of "the screening operator associated to $i \in I$" [Hernandez].

## Theorem (Nakajima, Hernandez)

$$\mathrm{ev}_{t=1}(K_t(\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}})) = \chi_q(K(\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}})).$$

# $(q,t)$-characters (1)

$\exists$ a $\mathbb{Z}[t^{\pm 1/2}]$-basis $\{M_t(m) \mid m \in \mathcal{B}\}$ of $K_t(\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}})$ such that

$$\mathrm{ev}_{t=1}(M_t(m)) = \chi_q(M(m)) \text{ [Nakajima, Hernandez].}$$

$\rightsquigarrow$ $M_t(m)$ is called the $(q,t)$-character of $M(m)$.

All $M_t(m)$ can be explicitly calculated once we know $M_t(Y_{i,0}), i \in I$.

**Theorem (Nakajima (ADE cases), Hernandez (arbitrary))**

$\exists ! \{L_t(m) \mid m \in \mathcal{B}\}$ *a* $\mathbb{Z}[t^{\pm 1/2}]$*-basis of* $K_t(\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}})$ *such that*

(S1) $\overline{L_t(m)} = L_t(m)$, *and*

(S2) $M_t(m) = L_t(m) + \sum_{m' < m} P_{m,m'}(t) L_t(m')$ *with* $P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$.

The element $L_t(m)$ is called the $(q,t)$-character of $L(m)$.

(S1) $\overline{L_t(m)} = L_t(m)$ (S2) $M_t(m) = L_t(m) + \sum_{m'<m} P_{m,m'}(t) L_t(m')$, $P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$

## Remark

The characterization properties (S1) and (S2) provide an inductive algorithm for computing $P_{m,m'}(t)$'s, called Kazhdan-Lusztig algorithm.

When $\mathfrak{g}$ is of $\mathrm{ADE}$ type,

$$\mathrm{ev}_{t=1}(L_t(m)) = \chi_q(L(m)) \text{ [Nakajima].}$$

Its proof is based on his geometric construction using graded quiver varieties, and it is valid only in $\mathrm{ADE}$ case. Moreover, in this case,

$$P_{m,m'}(t) \in t^{-1}\mathbb{Z}_{\geq 0}[t^{-1}] \text{ (positivity).}$$

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## Conjecture [Hernandez]

For arbitrary cases, we also have
(1) $\forall m \in \mathcal{B}$, $\mathrm{ev}_{t=1}(L_t(m)) = \chi_q(L(m))$. (2) $P_{m,m'}(t) \in t^{-1}\mathbb{Z}_{\geq 0}[t^{-1}]$.

If Conjecture (1) holds (in particular, in $\mathrm{ADE}$ cases), we have

$$[M(m)] = [L(m)] + \sum_{m'<m} P_{m,m'}(1)[L(m')] \text{ in } K(\mathcal{C}_{\bullet, \mathrm{X}_n^{(1)}}).$$

# Quantized coordinate algebra of type $\mathrm{A}_N$

Let $\mathcal{U}_v^-$ be the negative half of the QEA of type $\mathrm{A}_N$ over $\mathbb{Q}(v^{1/2})$.

$\left( := \text{the } \mathbb{Q}(v^{1/2})\text{-algebra with } \underline{\text{generators}}\ \{f_i\}_{i=1,\dots,N},\ \underline{\text{relations}} \begin{cases} f_i^2 f_j - (v+v^{-1}) f_i f_j f_i + f_j f_i^2 = 0 & \text{if } |i-j| = 1 \\ f_i f_j - f_j f_i = 0 & \text{if } |i-j| > 1. \end{cases} \right)$

$\rightsquigarrow \mathcal{A}_v[N_-^{\mathrm{A}_N}] \underset{\mathbb{Z}[v^{\pm 1/2}]\text{-subalg}}{\subset} \mathcal{U}_v^-$ the quantized coordinate algebra.

## Property

$$\mathbb{Q}(v^{\pm 1/2}) \otimes_{\mathbb{Z}[v^{\pm 1/2}]} \mathcal{A}_v[N_-^{\mathrm{A}_N}] \simeq \mathcal{U}_v^- \quad \mathbb{C} \otimes_{\mathbb{Z}[v^{\pm 1/2}]} \mathcal{A}_v[N_-^{\mathrm{A}_N}] \simeq \mathbb{C}[N_-^{\mathrm{A}_N}].$$

Here $N_-^{\mathrm{A}_N} := \{(N+1) \times (N+1)$ unipotent lower triangular matrices$\}$.

- $\exists \mathrm{ev}_{v=1} : \mathcal{A}_v[N_-^{\mathrm{A}_N}] \to \mathbb{C}[N_-^{\mathrm{A}_N}]$ a $\mathbb{Z}$-algebra homomorphism, called the specialization at $v = 1$.
- $\exists$ an $\mathbb{Z}$-algebra anti-involution $\sigma'$ on $\mathcal{A}_v[N_-^{\mathrm{A}_N}]$, called the (twisted) dual bar involution (e.g. $v^{1/2} \mapsto v^{-1/2}$).

($:=$ the restriction of the $\mathbb{Z}$-algebra anti-involution on $\mathcal{U}_v^-$ given by $v^{1/2} \mapsto v^{-1/2}$, $f_i \mapsto -f_i$.)

# Dual canonical bases

Let $\boldsymbol{i} = (i_1, i_2, \dots, i_\ell)$ be a reduced word of the longest element $w_0$ of the Weyl group $W^{\mathrm{A}_N} \simeq \mathfrak{S}_{N+1}$.
(e.g. if $N = 2$, then $\boldsymbol{i} = (1, 2, 1)$ or $(2, 1, 2)$. )

# Dual canonical bases

Let $\boldsymbol{i} = (i_1, i_2, \dots, i_\ell)$ be a reduced word of the longest element $w_0$ of the Weyl group $W^{\mathrm{A}_N} \simeq \mathfrak{S}_{N+1}$. Let $\Delta_+$ be the set of positive roots of type $\mathrm{A}_N$.
$\rightsquigarrow \exists\{\widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}) \mid \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+}\}$ a $\mathbb{Z}[v^{\pm 1/2}]$-basis of $\mathcal{A}_v[N_-^{\mathrm{A}_N}]$ depending on $\boldsymbol{i}$, which is an analogue of the (dual) PBW-basis associated to $\boldsymbol{i}$ [Lusztig]. By using this, we can characterize the (normalized) dual canonical basis $\widetilde{\mathbf{B}}^{\mathrm{up}}$ in the sense of Lusztig and Kashiwara.

## Theorem (Lusztig, Kimura)

- $\exists! \widetilde{\mathbf{B}}^{\mathrm{up}} := \{\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}) \mid \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+}\}$ *a* $\mathbb{Z}[v^{\pm 1/2}]$*-basis of* $\mathcal{A}_v[N_-^{\mathrm{A}_N}]$ *such that*
  - (B1) $\sigma'(\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i})) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i})$, *and*
  - (B2) $\widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}) + \sum_{\boldsymbol{c}'} p_{\boldsymbol{c},\boldsymbol{c}'}(v) \widetilde{G^{\mathrm{up}}}(\boldsymbol{c}', \boldsymbol{i})$ *with* $p_{\boldsymbol{c},\boldsymbol{c}'}(v) \in v\mathbb{Z}[v]$.
- $\widetilde{\mathbf{B}}^{\mathrm{up}}$ *does not depend on the choice of* $\boldsymbol{i}$.

# Positivities

(B1) $\sigma'(\widetilde{G^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i})) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i})$ (B2) $\widetilde{F^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i}) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i}) + \sum_{\boldsymbol{c}'} p_{\boldsymbol{c},\boldsymbol{c}'}(v)\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}',\boldsymbol{i}), p_{\boldsymbol{c},\boldsymbol{c}'}(v) \in v\mathbb{Z}[v]$

## Theorem (Lusztig ($\boldsymbol{i}$ "adapted"), Kato, (O.) (arbitrary)))

$$p_{\boldsymbol{c},\boldsymbol{c}'}(v) \in \mathbb{Z}_{\geq 0}[v].$$

## Theorem (Lusztig)

*For* $\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathbb{Z}_{\geq 0}^{\Delta_+}$, *write*

$$\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}_1,\boldsymbol{i})\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}_2,\boldsymbol{i}) = \sum_{\boldsymbol{c}} c^{\boldsymbol{c}}_{\boldsymbol{c}_1,\boldsymbol{c}_2}(v)\widetilde{G^{\mathrm{up}}}(\boldsymbol{c},\boldsymbol{i}).$$

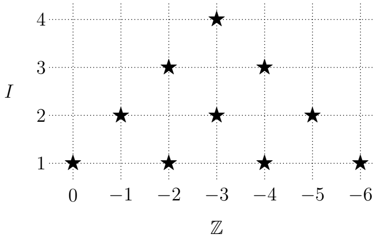
*Then* $c^{\boldsymbol{c}}_{\boldsymbol{c}_1,\boldsymbol{c}_2}(v) \in \mathbb{Z}_{\geq 0}[v^{\pm 1/2}]$.

# Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (1)

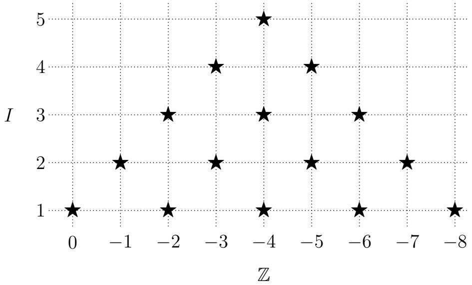
Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{A}_N^{(1)}$ ($I = \{1, \dots, N\}$).
Define $J_{\mathcal{Q},\mathrm{A}_N^{(1)}}$ by

$$J_{\mathcal{Q},\mathrm{A}_N^{(1)}} := \{(\imath, -\imath + 1 - 2k) \in I \times \mathbb{Z} \mid k = 0, 1, \dots, N - \imath \text{ and } \imath \in I\}.$$

$\underline{N = 4}$




$\underline{N = 5}$



# Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{A}_N^{(1)}$ $(I = \{1, \dots, N\})$.
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Set

$$\mathcal{B}_{\mathcal{Q},\mathrm{A}_N^{(1)}} := \left\{ \prod_{(\imath,r)} Y_{\imath,r}^{u_{\imath,r}} \in \mathcal{B} \,\middle|\, u_{\imath,r} \neq 0 \text{ only if } (\imath, r) \in J_{\mathcal{Q},\mathrm{A}_N^{(1)}} \right\},$$

$\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}}$ :=the full subcategory of $\mathcal{C}_{\bullet,\mathrm{A}_N^{(1)}}$ such that

<u>object</u> : $V$ with $[V] \in \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{A}_N^{(1)}}} \mathbb{Z}[L(m)]$.

## Lemma (Hernandez-Leclerc)

*$\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}}$ is an abelian $\otimes$-subcategory.*

# Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (2)

Set

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q},\,\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q},\,\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

## Lemma

$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}})$ *is a* $\mathbb{Z}[t^{\pm 1/2}]$*-subalgebra of* $K_t(\mathcal{C}_{\bullet,\mathrm{A}_n^{(1)}})$.

$\rightsquigarrow$ $K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}})$ is called the quantum Grothendieck ring of $\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}}$.

# Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (2)

Set

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}}) := \sum_{m\in\mathcal{B}_{\mathcal{Q},\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm1/2}]M_t(m) = \sum_{m\in\mathcal{B}_{\mathcal{Q},\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm1/2}]L_t(m).$$

Write

$J_{\mathcal{Q},\mathrm{A}_N^{(1)}} = \{(\imath_s, r_s) \mid s = 1,\ldots,\ell(= N(N+1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_\mathcal{Q} := (\imath_1, \imath_2, \ldots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_N}$.

**Remark**

The reduced word $\boldsymbol{i}_\mathcal{Q}$ is not uniquely determined, but its "commutation class" is well-defined. This $\boldsymbol{i}_\mathcal{Q}$ is "adapted to equioriented $\mathrm{A}_N$-quiver $\mathcal{Q}$".

# Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (2)

Set

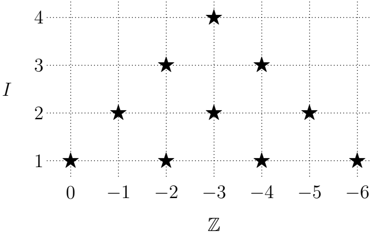
$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

$J_{\mathcal{Q},\mathrm{A}_N^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= N(N+1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathcal{Q}} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_N}$.

In the following example, $\boldsymbol{i}_{\mathcal{Q}} = (1, 2, 1, 3, 2, 4, 1, 3, 2, 1)$ etc.

$\underline{N = 4}$



# Hernandez-Leclerc isomorphisms in type $\mathrm{A}_N^{(1)}$ (2)

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathcal{Q},\mathrm{A}_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

$J_{\mathcal{Q},\mathrm{A}_N^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= N(N+1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathcal{Q}} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_N}$.

## Theorem (Hernandez-Leclerc)

*There exists a $\mathbb{Z}$-algebra isomorphism*

$$\Phi_{\mathrm{A}} \colon \mathcal{A}_v[N_-^{\mathrm{A}_N}] \xrightarrow{\sim} K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}})$$

*given by*

$$v^{\pm 1/2} \mapsto t^{\mp 1/2} \quad \widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathcal{Q}}) \mapsto M_t(m(\boldsymbol{c})) \ \forall \boldsymbol{c} = (c_\beta)_{\beta \in \Delta_+} \in \mathbb{Z}_{\geq 0}^{\Delta_+},$$

*here* $m(\boldsymbol{c}) = \prod_{k=1}^{\ell} Y_{\imath_k, r_k}^{c_{s_{\imath_1} \cdots s_{\imath_{k-1}} \alpha_{\imath_k}}}$. *Moreover,*

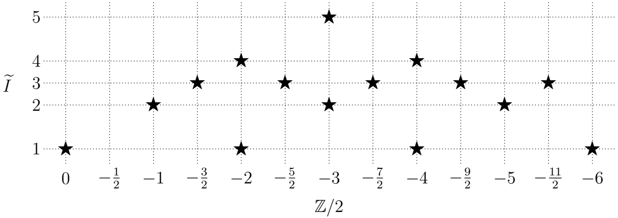
$$\Phi_{\mathrm{A}}(\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathcal{Q}})) = L_t(m(\boldsymbol{c})). \ \forall \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+}.$$

# Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}_n^{(1)}$ $(I = \{1, \ldots, n\})$.
Let $\widetilde{I} := \{1, \ldots, 2n-1\}$. Define $\widetilde{J}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$ by

$$\begin{aligned}\widetilde{J}_{\mathscr{Q},\mathrm{B}_n^{(1)}} :=& \{(\imath, -\imath + 2 - 2k) \mid k = 0, \ldots, 2n-1-\imath \text{ and } \imath = n+1, \ldots, 2n-1\} \\ & \cup \{(n, -n + \tfrac{3}{2} - k) \mid k = 0, \ldots, 2n-2\} \\ & \cup \{(\imath, -\imath + 1 - 2k) \mid k = 0, \ldots, 2n-2-\imath \text{ and } \imath = 1, \ldots, n-1\}.\end{aligned}$$

$\underline{n = 3}$

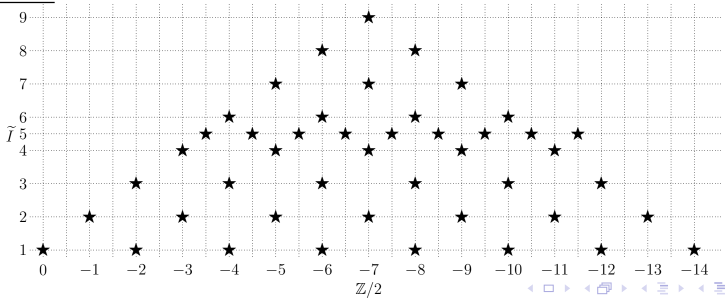


# Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}_n^{(1)}$ $(I = \{1, \dots, n\})$.
Let $\widetilde{I} := \{1, \dots, 2n-1\}$. Define $\widetilde{J}_{\mathscr{Q}, \mathrm{B}_n^{(1)}}$ by

$$\begin{aligned}\widetilde{J}_{\mathscr{Q}, \mathrm{B}_n^{(1)}} :=& \{(\imath, -\imath + 2 - 2k) \mid k = 0, \dots, 2n-1-\imath \text{ and } \imath = n+1, \dots, 2n-1\} \\ &\cup \{(n, -n + \tfrac{3}{2} - k) \mid k = 0, \dots, 2n-2\} \\ &\cup \{(\imath, -\imath + 1 - 2k) \mid k = 0, \dots, 2n-2-\imath \text{ and } \imath = 1, \dots, n-1\}.\end{aligned}$$

$\underline{n = 5}$



# Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}_n^{(1)}$ $(I = \{1, \dots, n\})$.
Let $\widetilde{I} := \{1, \dots, 2n-1\}$. Define $\widetilde{J}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$.

Consider the map $\tilde{I} \to I, \imath \mapsto \bar{\imath} := \begin{cases} \imath & \text{if } \imath \leq n, \\ 2n - \imath & \text{if } \imath > n. \end{cases}$ "folding"

Set

$$\mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}} := \left\{ \prod_{(i,r)} Y_{i,r}^{u_{i,r}} \in \mathcal{B} \;\middle|\; \begin{array}{l} u_{i,r} \neq 0 \text{ only if } (i,r) = (\bar{\imath}, 2s) \\ \text{for some } (\imath, s) \in \widetilde{J}_{\mathscr{Q},\mathrm{B}_n^{(1)}} \end{array} \right\},$$

$\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$ :=the full subcategory of $\mathcal{C}_\bullet$ such that

$$\underline{\text{object}} : V \text{ with } [V] \in \sum_{m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[L(m)].$$

**Lemma (Oh-Suh, Hernandez-O.)**

$\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$ *is an abelian* $\otimes$*-subcategory.*

# Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $\mathrm{B}_n^{(1)}$ ($I = \{1, \ldots, n\}$).
Let $\widetilde{I} := \{1, \ldots, 2n-1\}$.

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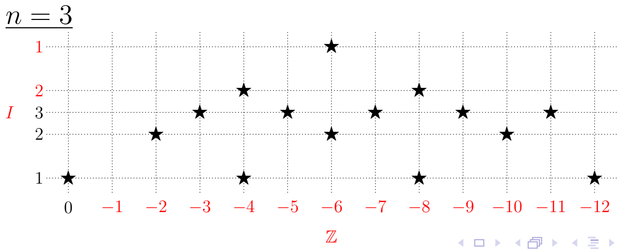


# Our isomorphisms (2)

Set

$$K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}) := \sum_{m\in\mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}]M_t(m) = \sum_{m\in\mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}]L_t(m).$$

**Lemma**

*$K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}})$ is a $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of $K_t(\mathscr{C}_{\bullet,\mathrm{B}_n^{(1)}})$.*

$\rightsquigarrow$ $K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}})$ is called the quantum Grothendieck ring of $\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$.

# Our isomorphisms (2)

Set

$$K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}) := \sum_{m\in\mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}]M_t(m) = \sum_{m\in\mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}]L_t(m).$$

Write

$\widetilde{J}_{\mathscr{Q},\mathrm{B}_n^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= 2n(2n-1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_{2n-1}}$.

## Remark

The reduced word $\boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}}$ is not uniquely determined, but its "commutation class" is well-defined. This $\boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}}$ is always "non-adapted".

# Our isomorphisms (2)

Set

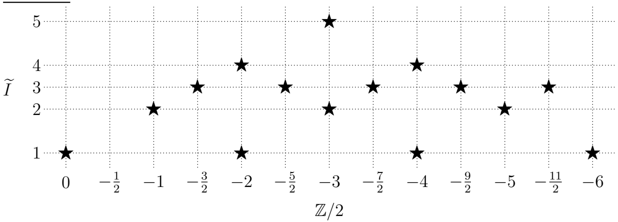
$$K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

$\widetilde{J}_{\mathscr{Q},\mathrm{B}_n^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= 2n(2n-1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_{2n-1}}$.

In the following example, $\boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}} = (1, 2, 3, 1, 4, 3, 2, 5, 3, 1, 4, 3, 2, 3, 1)$ etc.

$\underline{n = 3}$



# Our isomorphisms (2)

$$K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}) := \sum_{m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

$\widetilde{J}_{\mathscr{Q},\mathrm{B}_n^{(1)}} = \{(\imath_s, r_s) \mid s = 1, \dots, \ell(= 2n(2n-1)/2)\}$ with $r_1 \geq \cdots \geq r_\ell$.

$\rightsquigarrow \boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}} := (\imath_1, \imath_2, \dots, \imath_\ell)$ is a reduced word of $w_0 \in W^{\mathrm{A}_{2n-1}}$.

## Theorem (Hernandez-O.)

*There exists a $\mathbb{Z}$-algebra isomorphism*

$$\Phi_{\mathrm{B}} \colon \mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] \xrightarrow{\sim} K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}})$$

*given by*

$$v^{\pm 1/2} \mapsto t^{\mp 1/2} \quad \widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}}) \mapsto M_t(m^{\mathrm{tw}}(\boldsymbol{c})) \ \forall \boldsymbol{c} = (c_\beta)_{\beta \in \Delta_+} \in \mathbb{Z}_{\geq 0}^{\Delta_+},$$

*here* $m^{\mathrm{tw}}(\boldsymbol{c}) = \prod_{k=1}^{\ell} Y_{\overline{\imath_k}, 2r_k}^{c_{s_{\imath_1} \cdots s_{\imath_{k-1}} \alpha_{\imath_k}}}$. *Moreover,*

$$\Phi_{\mathrm{B}}(\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}})) = L_t(m^{\mathrm{tw}}(\boldsymbol{c})). \ \forall \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\Delta_+}.$$

# Positivities in $\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$

By our theorem, the positivities of the dual canonical bases $\widetilde{\mathbf{B}}^{\mathrm{up}}$ can be transported to those of $(q,t)$-characters.

## Corollary (Positivity of Kazhdan-Lusztig type polynomials)

*For $m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$, write*

$$M_t(m) = \sum_{m' \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} P_{m,m'}(t) L_t(m').$$

*as before. Then $P_{m,m'}(t) \in \mathbb{Z}_{\geq 0}[t^{-1}]$.*

This is the affirmative answer to Conjecture (2) for $\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$.

## Corollary (Positivity of structure constants)

*For $m_1, m_2 \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$, write*

$$L_t(m_1)L_t(m_2) = \sum_{m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}} c^m_{m_1,m_2}(t) L_t(m).$$

*Then we have $c^m_{m_1,m_2}(t) \in \mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$.*

# Comparison with Kashiwara-Oh

The following remarkable theorem is recently proved by means of the celebrated *generalized quantum affine Schur-Weyl dualities*, which is developed by Kang, Kashiwara, Kim and Oh :

## Theorem (Kashiwara-Oh '17)

*The correspondence* $\mathrm{ev}_{v=1}(\widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}})) \mapsto [M(m^{\mathrm{tw}}(\boldsymbol{c}))]$ *gives a* $\mathbb{Z}$*-algebra isomorphism*

$$[\mathscr{F}] \colon \mathrm{ev}_{v=1}(\mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}]) \xrightarrow{\sim} K(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}),$$

*which maps* $\mathrm{ev}_{v=1}(\widetilde{\mathbf{B}}^{\mathrm{up}})$ *to* $\{[L(m)] \mid m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}\}$.

## Theorem (Hernandez-O.)

$$\Phi_{\mathrm{B}} \mid_{v=t=1} = [\mathscr{F}].$$

# Comparison with Kashiwara-Oh

## Theorem (Hernandez-O.)

$$\Phi_{\mathrm{B}}\,|_{v=t=1} = [\mathscr{F}].$$

## Remark

Our construction of $\Phi_{\mathrm{B}}$ does not imply Kashiwara-Oh's theorem because, a priori,

- $\Phi_{\mathrm{B}}|_{v=t=1}$ maps $\mathrm{ev}_{v=1}(\widetilde{\mathbf{B}}^{\mathrm{up}})$ to $\{\mathrm{ev}_{t=1}(L_t(m)) | m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}\}$, but
- $[\mathscr{F}]$ maps $\mathrm{ev}_{v=1}(\widetilde{\mathbf{B}}^{\mathrm{up}})$ to $\{[L(m)] \mid m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}\}$,

(The coincidence of these images is nothing but Hernandez's conjecture (1) !) Hence our result and Kashiwara-Oh's result are independent.

# Comparison with Kashiwara-Oh

## Theorem (Kashiwara-Oh '17)

*The correspondence* $\mathrm{ev}_{v=1}(\widetilde{F^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}^{\mathrm{tw}}_{\mathscr{Q}})) \mapsto [M(m^{\mathrm{tw}}(\boldsymbol{c}))]$ *gives a* $\mathbb{Z}$*-algebra isomorphism*

$$[\mathscr{F}]\colon \mathrm{ev}_{v=1}(\mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}]) \xrightarrow{\sim} K(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}),$$

*which maps* $\mathrm{ev}_{v=1}(\widetilde{\mathbf{B}}^{\mathrm{up}})$ *to* $\{[L(m)] \mid m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}\}$.

## Theorem (Hernandez-O.)

$$\Phi_{\mathrm{B}} \,|_{v=t=1} = [\mathscr{F}].$$

## Corollary

$$\mathrm{ev}_{t=1}(L_t(m)) = \chi_q(L(m)), \forall m \in \mathcal{B}_{\mathscr{Q},\mathrm{B}_n^{(1)}}.$$

This is the affirmative answer to Conjecture (1) for $\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$.

- There are several variants in the choices of the subcategories $\mathcal{C}_{\mathcal{Q},\mathrm{A}_N^{(1)}}$ and $\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}$, and the parallel results hold. (The choice in this talk is the case that $\mathcal{Q}$ and $\mathscr{Q}$ are "equioriented".)
- By $\Phi_{\mathrm{A}}$ and $\Phi_{\mathrm{B}}$, we can obtain $\mathbb{Z}[v^{\pm 1/2}]$-algebra isomorphism :

**Theorem (Hernandez-O.)**

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_{2n-1}^{(1)}}) \simeq K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}),\ \left\{\begin{array}{c}(q,t)\text{-characters of}\\ \text{simple modules}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}(q,t)\text{-characters of}\\ \text{simple modules}\end{array}\right\}.$$

*Moreover, we can calculate the explicit correspondence of simple modules.*

To calculate the correspondence, we have to calculate the pair $(\boldsymbol{c}, \boldsymbol{c}')$ such that

$$\widetilde{G^{\mathrm{up}}}(\boldsymbol{c}, \boldsymbol{i}_{\mathcal{Q}}) = \widetilde{G^{\mathrm{up}}}(\boldsymbol{c}', \boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}})$$

# Comments on further results and proofs (2)

e.g. $\mathrm{A}_5^{(1)}/\mathrm{B}_3^{(1)}$-correspondence ($n=3$) :

$$
\begin{aligned}
&L_t^{\mathrm{A}}(Y_{1,0}) \leftrightarrow L_t^{\mathrm{B}}(Y_{1,0}), && L_t^{\mathrm{A}}(Y_{1,-2}) \leftrightarrow L_t^{\mathrm{B}}(Y_{1,-4}),\\
&L_t^{\mathrm{A}}(Y_{1,-4}) \leftrightarrow L_t^{\mathrm{B}}(Y_{3,-11}), && L_t^{\mathrm{A}}(Y_{1,-6}) \leftrightarrow L_t^{\mathrm{B}}(Y_{3,-5}),\\
&L_t^{\mathrm{A}}(Y_{1,-8}) \leftrightarrow L_t^{\mathrm{B}}(Y_{1,-12}), && L_t^{\mathrm{A}}(Y_{2,-1}) \leftrightarrow L_t^{\mathrm{B}}(Y_{2,-2}),\\
&L_t^{\mathrm{A}}(Y_{2,-3}) \leftrightarrow L_t^{\mathrm{B}}(Y_{3,-7}), && L_t^{\mathrm{A}}(Y_{2,-5}) \leftrightarrow L_t^{\mathrm{B}}(Y_{3,-5}Y_{3,-11}),\\
&L_t^{\mathrm{A}}(Y_{2,-7}) \leftrightarrow L_t^{\mathrm{B}}(Y_{3,-9}), && L_t^{\mathrm{A}}(Y_{3,-2}) \leftrightarrow L_t^{\mathrm{B}}(Y_{3,-3}),\\
&L_t^{\mathrm{A}}(Y_{3,-4}) \leftrightarrow L_t^{\mathrm{B}}(Y_{3,-5}Y_{3,-7}), && L_t^{\mathrm{A}}(Y_{3,-6}) \leftrightarrow L_t^{\mathrm{B}}(Y_{3,-9}Y_{3,-11}),\\
&L_t^{\mathrm{A}}(Y_{4,-3}) \leftrightarrow L_t^{\mathrm{B}}(Y_{2,-4}), && L_t^{\mathrm{A}}(Y_{4,-5}) \leftrightarrow L_t^{\mathrm{B}}(Y_{2,-8}),\\
&L_t^{\mathrm{A}}(Y_{5,-4}) \leftrightarrow L_t^{\mathrm{B}}(Y_{1,-6}).
\end{aligned}
$$

It does not preserve dimension and “degree” of highest weight !

# Comments on further results and proofs (3)

Sketch of the proof of the existence of $\Phi_{\mathrm{B}}$

0) We have

- $K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}}) \overset{\text{“truncate”}}{\hookrightarrow}$ the quantum torus of *finitely many* variables.
- $\mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}] \hookrightarrow$ the quantum torus arising from the “quantum initial seed” associated with $\boldsymbol{i}_{\mathscr{Q}}^{\mathrm{tw}}$ ($\Leftarrow$ quantum cluster algebra).

1) Prove the isomorphism between ambient tori in Step 0. (Here we also use the cluster algebraic observation “$A_{i,r}^{-1}$'s are $\hat{y}$-variables”)

2) Prove the coincidence between quantum $T$-system and quantum determinantal ientities via the isomorphism 1)
$\rightsquigarrow$ We can show the correspondence between generators of $K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}})$ and $\mathcal{A}_v[N_-^{\mathrm{A}_{2n-1}}]$ !

Reference : arXiv:1803.06754

# $T$-system

For $i \in I$, $r \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$, set $m^{(i)}_{k,r} := \prod_{s=1}^{k} Y_{i,r+2r_i(s-1)}$. $(m^{(i)}_{1,r} = Y_{i,r})$

## The quantum $T$-system of type $\mathrm{B}$ [Hernandez-O.]

$\exists \alpha, \beta \in \mathbb{Z}$ *such that the following identity holds in* $K_t(\mathcal{C}_{\mathscr{Q},\mathrm{B}_n^{(1)}})$ *:*

$$L_t(m^{(i)}_{k,r})L_t(m^{(i)}_{k,r+2r_i}) = t^{\alpha/2}L_t(m^{(i)}_{k+1,r})L_t(m^{(i)}_{k-1,r+2r_i}) + t^{\beta/2}S^{(i)}_{k,r,t}.$$

*Here,* $S^{(i)}_{k,r,t} = \begin{cases} L_t(m^{(i-1)}_{k,r+2})L_t(m^{(i+1)}_{k,r+2}) & \textit{if } i \leq n-2, \\ L_t(m^{(n-2)}_{k,r+2})L_t(m^{(n)}_{2k,r+1}) & \textit{if } i = n-1, \\ L_t(m^{(n-1)}_{s,r+1})L_t(m^{(n-1)}_{s,r+3}) & \textit{if } i = n \textit{ and } k = 2s \textit{ is even}, \\ L_t(m^{(n-1)}_{s+1,r+1})L_t(m^{(n-1)}_{s,r+3}) & \textit{if } i = n \textit{ and } k = 2s+1 \textit{ is odd}. \end{cases}$ $(L_t(m^{(0)}_{*,*}) := 1)$.

## Example ($\mathrm{B}_3^{(1)}$-case)

- $L_t(m^{(1)}_{2,r})L_t(m^{(1)}_{2,r+4}) = tL_t(m^{(1)}_{3,r})L_t(m^{(1)}_{1,r+4}) + L_t(m^{(2)}_{2,r+2})$.
- $L_t(m^{(3)}_{3,r})L_t(m^{(3)}_{3,r+2}) = t^{1/2}L_t(m^{(3)}_{4,r})L_t(m^{(3)}_{2,r+2}) + t^{-1/2}L_t(m^{(2)}_{2,r+1})L_t(m^{(2)}_{1,r+3})$.