Twist maps on quantum unipotent cells and the Chamber Ansatz

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Introduction

Aims of this talk:

Establish a quantum analogue of "the Chamber Ansatz".

Classical ($q = 1$) Factorization problem and the Chamber Ansatz:

Consider the following map:

$$\begin{array}{cccc} y_{\boldsymbol{i}}\colon & (\mathbb{C}^\times)^\ell & \to & N_-^w \\ & \Cup & & \Cup \\ & (t_1, \dots, t_\ell) & \longmapsto & \exp(t_1 F_{i_1}) \cdots \exp(t_\ell F_{i_\ell}). \end{array}$$

Here $\boldsymbol{i}$ a reduced word of w, and $N_-^w := N_- \cap B_+ w B_+$ unipotent cell. (In fact, This gives a birational isomorphism from $\mathbb{C}^\ell$ to a Schubert cell $X(w)$.)

Problem

Describe the inverse birational isomorphism $y_{\boldsymbol{i}}^{-1}$.

Introduction (2)

Berenstein-Zelevinsky (1997) gives formulae for $y_{\boldsymbol{i}}^{-1}$, called "the Chamber Ansatz". The key tool is a twist map $\eta_w^*\colon \mathbb{C}[N_-^w] \to \mathbb{C}[N_-^w]$. By the way, there are known q-analogues of $\mathbb{C}[N_-^w]$ and $y_{\boldsymbol{i}}$. The following are the main result.

Theorem (Kimura-O)

There exists "q-analogue" of the twist map η_w^*. Moreover quantum twist maps preserve dual canonical bases.

Theorem (O)

The Chamber Ansatz formulae also hold in quantum settings by using quantum twist maps above.

The Chamber Ansatz

Let

- $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ triangular decomposition (fixed),
- $\{E_i, F_i, H_i \mid i \in I\}$ Chevalley generators of $\mathfrak{g}$, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix (i.e. $[H_i, E_j] = a_{ij}E_j, \dots$),
- G connected simply connected algebraic group (over $\mathbb{C}$) with $\mathrm{Lie}\, G = \mathfrak{g}$,
- N_-, H, N_+ closed subgroups of G such that $\mathrm{Lie}\, N_- = \mathfrak{n}^-$, $\mathrm{Lie}\, H = \mathfrak{h}$, $\mathrm{Lie}\, N_+ = \mathfrak{n}^+$,
- $B_- := N_-H$, $B_+ := HN_+$ Borel subgroups,
- $x_i(t) = \exp(tE_i)$, $y_i(t) = \exp(tF_i)$ 1-parameter subgroups corresponding to E_i, F_i,
- $W := N_G(H)/H$ Weyl group, e its unit, $\{s_i \mid i \in I\}$ simple reflections, $\ell(w)$ the length of $w \in W$,

The Chamber Ansatz

Let $\mathfrak{g}$, G, $N_\pm$, H, $B_\pm$, $x_i(t)$, $y_i(t)$, W standard notation.

- $I(w) := \{(i_1, \dots, i_{\ell(w)}) \in I^{\ell(w)} \mid w = s_{i_1} \cdots s_{i_{\ell(w)}}\}$ the set of reduced words of $w \in W$,
- $\overline{s}_i := x_i(-1)y_i(1)x_i(-1)$ $(i \in I)$, $\overline{w} := \overline{s}_{i_1} \cdots \overline{s}_{i_\ell}$ $((i_1, \dots, i_\ell) \in I(w))$,
  In fact, $\overline{w}$ does not depend on the choice of $(i_1, \dots, i_\ell) \in I(w)$,
- $\varpi_i \in \mathrm{Hom}_{\mathrm{alg.grp.}}(H, \mathbb{C}^\times)$ fundamental weight corresponding to $i \in I$,
- $G_0 := N_- H N_+$, $g = [g]_-[g]_0[g]_+$ $(g \in G_0)$ the corresponding decomposition,

The Chamber Ansatz

Let $\mathfrak{g}$, G, $N_{\pm}$, H, $B_{\pm}$, $x_i(t)$, $y_i(t)$, W, $I(w)$, $\overline{w}$, ϖ_i standard notation. Set $G_0 := N_- H N_+$, $g = [g]_-[g]_0[g]_+$ $(g \in G_0)$.

Definition (Generalized minors)

For $i \in I$, denote by $\Delta_{\varpi_i,\varpi_i}$ the regular function on G whose restriction to the open dense set G_0 is given by

$$\Delta_{\varpi_i,\varpi_i}(g) := \varpi_i([g]_0)$$

For $w_1, w_2 \in W$, define $\Delta_{w_1\varpi_i,w_2\varpi_i} \in \mathbb{C}[G]$ by

$$\Delta_{w_1\varpi_i,w_2\varpi_i}(g) = \Delta_{\varpi_i,\varpi_i}(\overline{w_1}^{-1} g \overline{w_2})$$

These elements are called generalized minors.

The Chamber Ansatz (2)

For $w \in W$, set $N_-^w := N_- \cap B_+ \bar{w} B_+$ unipotent cell.

Fact (Twist maps [Berenstein-Zelevinsky])

We can define a biregular isomorphism $\eta_w \colon N_-^w \to N_-^w$ *by*

$$\eta_w(z) := [z^T \overline{w}]_-.$$

Recall the map

$$\begin{array}{cccc} y_{\boldsymbol{i}} \colon & (\mathbb{C}^\times)^\ell & \to & N_-^w \\ & \cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid \\ & (t_1, \dots, t_\ell) & \longmapsto & y_{i_1}(t_1) \cdots y_{i_\ell}(t_\ell). \end{array}$$

Here $\boldsymbol{i} = (i_1, \dots i_\ell) \in I(w)$.

The Chamber Ansatz (2)

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Fact (Twist maps [Berenstein-Zelevinsky])

We can define a biregular isomorphism $\eta_w \colon N_-^w \to N_-^w$ *by*

$$\eta_w(z) := [z^T \overline{w}]_-.$$

Theorem (Berenstein-Zelevinsky)

Let $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$. *For* $k \in \{1, \ldots, \ell\}$, *set* $w_{\leq k} := s_{i_1} \cdots s_{i_k}$. *Set* $y = y_{\boldsymbol{i}}(t_1, \ldots, t_\ell)$. *Then*

$$t_k = \frac{\prod_{j \in I \setminus \{i_k\}} \Delta_{w_{\leq k}\varpi_j, \varpi_j}(\eta_w^{-1}(y))^{-a_{j,i_k}}}{\Delta_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}}(\eta_w^{-1}(y)) \Delta_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}(\eta_w^{-1}(y))}$$

for $k \in \{1, \ldots, \ell\}$.

This formula is called the Chamber Ansatz.

q-analogue

From now on, we consider a q-analogue of the theorem above. In the settings of q-analogues, we do not have "actual group" but only have "coordinate rings". Hence we should consider the problem above in terms of coordinate rings.

The map $y_{\boldsymbol{i}}^*$ induces an injective algebra homomorphism

$$y_{\boldsymbol{i}}^* \colon \mathbb{C}[N_-^w] \to \mathbb{C}[t_1^{\pm 1}, \ldots, t_\ell^{\pm}].$$

The twist map η_w induces the algebra automorphism

$$\eta_w^* \colon \mathbb{C}[N_-^w] \to \mathbb{C}[N_-^w].$$

The q-analogue of the former is known as a Feigin homomorphism (explained later). Moreover, by using (the restriction of) generalized minors of the form $\Delta_{w'\varpi_i,\varpi_i}$, we can easily check the following formula;

$$\eta_w^*(\Delta_{w'\varpi_i,\varpi_i}) = \Delta_{w\varpi_i,\varpi_i}^{-1}\Delta_{w\varpi_i,w'\varpi_i}.$$

Note: $\Delta_{w'\varpi_i,\varpi_i}$, $\Delta_{w\varpi_i,\varpi_i}^{-1}\Delta_{w\varpi_i,w'\varpi_i} \in$ (dual canonical bases).

Setup

Notation

Let

- $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ a symmetrizable Kac-Moody Lie algebra($\supset$ finite dimensional simple Lie algebra) over $\mathbb{C}$ with (fixed) triangular decomposition,
- $\{\alpha_i\}_{i\in I}$ the simple roots of $\mathfrak{g}$, $\{h_i\}_{i\in I}$ the simple coroots of $\mathfrak{g}$,
- P a $\mathbb{Z}$-lattice (weight lattice) of $\mathfrak{h}^*$ and $P^* := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \subset \mathfrak{h}$ such that $\{\alpha_i\}_{i\in I} \subset P$ and $\{h_i\}_{i\in I} \subset P^*$,
- $P_+ := \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in I\}$. Set $\langle \varpi_i, h_j \rangle = \delta_{ij}$.
- W the Weyl group of $\mathfrak{g}$ ($W \curvearrowright P, P^*$),
- $I(w)$ the set of reduced words of $w \in W$,
- $(-,-) : P \times P \to \mathbb{Q}$ a $\mathbb{Q}$-valued (W-invariant) symmetric $\mathbb{Z}$-bilinear form on P satisfying the following conditions: $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$, $\langle \lambda, h_i \rangle = 2\,(\lambda, \alpha_i)\,/\,(\alpha_i, \alpha_i)$ for $i \in I$, $\lambda \in P$.

Quantized enveloping algebras

Definition

The quantized enveloping algebra $\mathbf{U}_q (:= \mathbf{U}_q(\mathfrak{g}))$ over $\mathbb{Q}(q)$ is the $\mathbb{Q}(q)$-algebra generated by

$$e_i, f_i \ (i \in I), \ q^h \ (h \in P^*),$$

with the following relations:
(i) $q^0 = 1, \ q^h q^{h'} = q^{h+h'}$,
(ii) $q^h e_i = q^{\langle h, \alpha_i \rangle} e_i q^h, \ q^h f_i = q^{-\langle h, \alpha_i \rangle} f_i q^h$,
(iii) $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ where $q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}$ and $t_i := q^{\frac{(\alpha_i, \alpha_i)}{2} h_i}$,
(iv) $\sum_{k=0}^{1-\langle h_i, \alpha_j \rangle} (-1)^k x_i^{(k)} x_j x_i^{(1-\langle h_i, \alpha_j \rangle - k)} = 0$ for $i \neq j$ $(x = e, f)$,
where $x_i^{(n)} := x_i^n / [n]_i!$, $[n]_i! := \prod_{k=1}^n (q_i^k - q_i^{-k})/(q_i - q_i^{-1})$.

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Relations: $q^h e_i = q^{\langle \alpha_i, h \rangle} e_i q^h$, q-Serre relations, ...
Let $\mathbf{U}_q^-$ be the subalgebra of $\mathbf{U}_q$ generated by f_i's.

Hopf algebra structure of $\mathbf{U}_q$

$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \ \Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i, \ \Delta(q^h) = q^h \otimes q^h,$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \varepsilon(q^h) = 1, \exists \text{antipode } S.$$

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Relations: $q^h e_i = q^{\langle \alpha_i, h \rangle} e_i q^h$, q-Serre relations, ...
Let $\mathbf{U}_q^-$ be the subalgebra of $\mathbf{U}_q$ generated by f_i's.

Hopf algebra structure of $\mathbf{U}_q$ (Δ, ε, S)
Let $^-\colon \mathbf{U}_q \to \mathbf{U}_q$ be the $\mathbb{Q}$-algebra involution defined by

$$\overline{q} = q^{-1}, \qquad \overline{e_i} = e_i, \qquad \overline{f_i} = f_i, \qquad \overline{q^h} = q^{-h}.$$

Let $(-)^T\colon \mathbf{U}_q \to \mathbf{U}_q$ be the $\mathbb{Q}(q)$-algebra anti-involutions defined by

$$(e_i)^T = f_i, \qquad (f_i)^T = e_i, \qquad (q^h)^T = q^h.$$

Canonical bases

Review the theory of canonical bases due to Lusztig and Kashiwara: Denote by $\mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]}$ the $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$ generated by the elements $\{f_i^{(n)} \mid i \in I, n \in \mathbb{Z}_{\geq 0}\}$. Then there exists a free $\mathcal{A}_0$-submodule $\mathscr{L}(\infty)$ of $\mathbf{U}_q^-$ such that

$$\begin{array}{ccc} \mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]} \cap \mathscr{L}(\infty) \cap \overline{\mathscr{L}(\infty)} & \xrightarrow{\text{projection}} & \mathscr{L}(\infty)/q\mathscr{L}(\infty) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{low}} := \{G^{\mathrm{low}}(b) \mid b \in \mathscr{B}(\infty)\} & \longmapsto & \mathscr{B}(\infty) \end{array}$$

is the isomorphism of $\mathbb{Q}$-vector spaces. Moreover we can construct a "special" $\mathbb{Q}$-basis $\mathscr{B}(\infty)$ of $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$. The inverse image of $\mathscr{B}(\infty)$ under this map is called the canonical bases $\mathbf{B}^{\mathrm{low}}$ of $\mathbf{U}_q^-$. In fact, $\mathbf{B}^{\mathrm{low}} = \{G^{\mathrm{low}}(b) \mid b \in \mathscr{B}(\infty)\}$ is a $\mathbb{Q}[q^{\pm 1}]$-basis of $\mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]}$.
For $b \in \mathscr{B}(\infty)$, $\overline{G^{\mathrm{low}}(b)} = G^{\mathrm{low}}(b)$ (bar-invariance property).

Dual canonical bases

Definition

There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_L \colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$ such that

$$(1,1)_L = 1, \qquad (f_i x, y)_L = \frac{1}{1-q_i^2}(x, e_i'(y))_L.$$

where $e_i' \colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ is the $\mathbb{Q}(q)$-linear map given by

$$e_i'(xy) = e_i'(x)\, y + q_i^{\langle \mathrm{wt}\, x, h_i \rangle} x e_i'(y), \quad e_i'(f_j) = \delta_{ij},$$

for homogeneous elements $x, y \in \mathbf{U}_q^-$.

Dual canonical bases

Definition

There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_L \colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$.
Denote by $\mathbf{B}^{\mathrm{up}}$ the basis of $\mathbf{U}_q^-$ dual to $\mathbf{B}^{\mathrm{low}}$ with respect to the bilinear form $(\ ,\ )_L$, that is, $\mathbf{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(b) \mid b \in \mathscr{B}(\infty)\}$ such that

$$(G^{\mathrm{low}}(b), G^{\mathrm{up}}(b'))_L = \delta_{b,b'} \text{ for } b, b' \in \mathscr{B}(\infty).$$

Definition (The dual bar-involution)

Define $\mathbb{Q}$-linear map $\sigma \colon \mathbf{U}_q^- \to \mathbf{U}_q^-, x \mapsto \sigma(x) = \sigma_L(x)$ by

$$(\sigma(x), y)_L = \overline{(x, \overline{y})_L} \text{ for arbitrary } y \in \mathbf{U}_q^-.$$

For $b \in \mathscr{B}(\infty)$, $\sigma(G^{\mathrm{up}}(b)) = G^{\mathrm{up}}(b)$ (dual bar-invariance property).

Specialization

Set

$$\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] := \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]})_L \in \mathbb{Q}[q^{\pm 1}]\} = \sum_{b \in \mathscr{B}(\infty)} \mathbb{Q}[q^{\pm 1}] G^{\mathrm{up}}(b).$$

Then $\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-]$ is a $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$.
Specialization:

$$\mathbf{U}_q^- \supset \mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]} \xrightarrow[\mathbb{C}\otimes_{\mathbb{Q}[q^{\pm 1}]}-]{\text{“}q\to 1\text{”}} \mathbf{U}(\mathfrak{n}^-)$$

$$\mathbf{U}_q^- \supset \mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] \xrightarrow[\mathbb{C}\otimes_{\mathbb{Q}[q^{\pm 1}]}-]{\text{“}q\to 1\text{”}} (\mathbf{U}(\mathfrak{n}^-))^*_{\mathrm{gr}} \simeq \mathbb{C}[N_-].$$

Here $(\mathbf{U}(\mathfrak{n}^-))^*_{\mathrm{gr}}$ denotes the graded dual of $\mathbf{U}(\mathfrak{n}^-)$. Hence we can regard $\mathbf{U}_q^-$ as a q-analogue of the coordinate ring $\mathbb{C}[N_-]$ if we take the dual canonical basis into account.

Quantum closed unipotent cell

Proposition (Kashiwara)

For $w \in W$ and $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$, set

$$\mathbf{U}_{q,w}^- := \sum_{a_1,\cdots,a_\ell} \mathbb{Q}(q)\, f_{i_1}^{a_1} \cdots f_{i_\ell}^{a_\ell}.$$

Then the following hold:

(1) *The subspace $\mathbf{U}_{q,w}^-$ does not depend on the choice of $\boldsymbol{i} \in I(w)$.*

(2) *Set $(\mathbf{U}_{q,w}^-)^\perp := \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}_{q,w}^-)_L = 0\}$. Then $(\mathbf{U}_{q,w}^-)^\perp$ is a two-sided ideal of $\mathbf{U}_q^-$.*

(3) *$(\mathbf{U}_{q,w}^-)^\perp \cap \mathbf{B}^{\mathrm{up}}$ is a basis of $(\mathbf{U}_{q,w}^-)^\perp$ (equivalently, $\mathbf{U}_{q,w}^- \cap \mathbf{B}^{\mathrm{low}}$ is a basis of $\mathbf{U}_{q,w}^-$).*

Quantum closed unipotent cell (2)

Definition (Quantum closed unipotent cell)

For $w \in W$, set

$$\mathbf{A}_q[\overline{N_-^w}] := \mathbf{U}_q^- / (\mathbf{U}_{q,w}^-)^\perp.$$

This is an algebra, called a quantum closed unipotent cell, by the proposition above.

By the proposition (3) above, the subset of $\mathbf{B}^{\mathrm{up}}$ induces a basis of $\mathbf{A}_q[\overline{N_-^w}]$. Denote by $\mathscr{B}_w(\infty)$ the corresponding subset of $\mathscr{B}(\infty)$ (called a Demazure crystal). The natural projection $\mathbf{U}_q^- \to \mathbf{A}_q[\overline{N_-^w}]$ will be described as $x \mapsto \underline{x}$. In fact, we have

$$\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[\overline{N_-^w}] := \sum_{b \in \mathscr{B}_w(\infty)} \mathbb{Q}[q^{\pm 1}]\underline{G^{\mathrm{up}}(b)} \xrightarrow[\mathbb{C}\otimes_{\mathbb{Q}[q^{\pm 1}]}-]{\text{“}q\to 1\text{”}} \mathbb{C}[\overline{N_-^w}].$$

Unipotent quantum minors

For $\lambda \in P_+$, denote by $V(\lambda)$ the integrable highest weight $\mathbf{U}_q$-module generated by a highest weight vector u_λ of weight λ. For $w \in W$ and $\boldsymbol{i} \in I(w)$, set

$$u_{w\lambda} = f_{i_1}^{(\langle h_{i_1}, s_{i_2} \cdots s_{i_\ell} \lambda \rangle)} \cdots f_{i_{\ell-1}}^{(\langle h_{i_{\ell-1}}, s_{i_\ell} \lambda \rangle)} f_{i_\ell}^{(\langle h_{i_\ell}, \lambda \rangle)} . u_\lambda .$$

There exists a unique nondegenerate and symmetric bilinear form $(\ ,\ )_\lambda \colon V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ such that

$$(u_\lambda, u_\lambda)_\lambda = 1 \qquad (x.v_1, v_2)_\lambda = (v_1, x^T.v_2)_\lambda$$

for $v_1, v_2 \in V(\lambda)$ and $x \in \mathbf{U}_q$.

Definition (Unipotent quantum minors)

For $\lambda \in P_+$ and $v_1, v_2 \in V(\lambda)$, define an element $D_{v_1,v_2} \in \mathbf{U}_q^-$ by

$$(D_{v_1,v_2}, x)_L = (v_1, x.v_2)_\lambda \text{ for arbitrary } x \in \mathbf{U}_q^- .$$

For $w_1, w_2 \in W$, write $D_{w_1\lambda, w_2\lambda} := D_{u_{w_1\lambda}, u_{w_2\lambda}}$.

Quantum unipotent cell

Proposition (Kashiwara)

For $\lambda \in P_+$, $w_1, w_2 \in W$, *we have* $D_{w_1\lambda, w_2\lambda} \in \mathbf{B}^{\mathrm{up}} \coprod \{0\}$.

Proposition

Let $w \in W$. *Then* $\underline{\mathcal{D}_w} := q^{\mathbb{Z}}\{\underline{D_{w\lambda,\lambda}}\}_{\lambda \in P_+}$ *is an Ore set of* $\mathbf{A}_q[\overline{N_-^w}]$ *consisting of* q*-central elements.*

Definition

For $w \in W$, we can consider the algebras of fractions

$$\mathbf{A}_q[N_-^w] := \mathbf{A}_q[\overline{N_-^w}][\underline{\mathcal{D}_w^{-1}}]$$

by the proposition above. This algebra is called a quantum unipotent cell.

Proposition

Let $w \in W$. Then

$$\tilde{\mathbf{B}}_w^{\mathrm{up}} := \{q^{(\lambda, \mathrm{wt}\, b + \lambda - w\lambda)} \underline{D_{w\lambda,\lambda}}^{-1} \underline{G^{\mathrm{up}}(b)} \mid \lambda \in P_+, b \in \mathscr{B}_w(\infty)\}$$

forms a basis of $\mathbf{A}_q[N_-^w]$. We call $\tilde{\mathbf{B}}_w^{\mathrm{up}}$ the dual canonical bases of $\mathbf{A}_q[N_-^w]$.

The dual bar involution σ on $\mathbf{U}_q^-$ induces the $\mathbb{Q}$-linear isomorphism $\sigma\colon \mathbf{A}_q[\overline{N_-^w}] \to \mathbf{A}_q[\overline{N_-^w}]$, and this is extended to the $\mathbb{Q}$-linear isomorphism $\sigma\colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w]$ satisfying

$$\sigma(xy) = q^{(\mathrm{wt}\, x, \mathrm{wt}\, y)} \sigma(y)\sigma(x)$$

for homogeneous elements $x, y \in \mathbf{A}_q[N_-^w]$ (We can naturally define the Q-graded structure on $\mathbf{A}_q[N_-^w]$).
Then every element of $\tilde{\mathbf{B}}_w^{\mathrm{up}}$ is fixed by σ.

Quantum twist maps (2)

Theorem (Kimura-O)

Let $w \in W$. Then there exists the automorphism of the $\mathbb{Q}(q)$-algebra

$$\eta_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w],$$

given by

$$\underline{D_{v,u_\lambda}} \mapsto q^{-(\lambda, \mathrm{wt}\, v - \lambda)} \underline{D_{w\lambda,\lambda}}^{-1} \underline{D_{u_{w\lambda},v}}$$

for all $\lambda \in P_+$ and weight vectors $v \in V(\lambda)$. Moreover $\eta_{w,q}$ is restricted to the permutation of $\tilde{\mathrm{B}}_w^{\mathrm{up}}$.

We call $\eta_{w,q}$ a quantum twist map. For example we have

$$\eta_{w,q}(\underline{D_{w'\varpi_i,\varpi_i}}) = q^{-(\varpi_i, w'\varpi_i - \varpi_i)} \underline{D_{w\varpi_i,\varpi_i}}^{-1} \underline{D_{w\varpi_i,w'\varpi_i}}.$$

(cf. $\eta_w^*(\Delta_{w'\varpi_i,\varpi_i}) = \Delta_{w\varpi_i,\varpi_i}^{-1} \Delta_{w\varpi_i,w'\varpi_i}$.)

Feigin homomorphisms

Definition (Feigin homomorphisms)

Let $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I^\ell$. The Laurent q-polynomial algebra $L_{\boldsymbol{i}}$ is the unital associative $\mathbb{Q}(q)$-algebra generated by $t_1^{\pm 1}, \ldots, t_\ell^{\pm 1}$ subject to the relations;

$$t_j t_k = q^{(\alpha_{i_j}, \alpha_{i_k})} t_k t_j \text{ for } 1 \leq j < k \leq \ell,$$
$$t_k t_k^{-1} = t_k^{-1} t_k = 1 \text{ for } 1 \leq k \leq \ell.$$

Then we can define the $\mathbb{Q}(q)$-linear map $\Phi_{\boldsymbol{i}} \colon \mathbf{U}_q^- \to L_{\boldsymbol{i}}$ by

$$x \mapsto \sum_{\boldsymbol{a}=(a_1,\ldots,a_\ell) \in \mathbb{Z}_{\geq 0}^\ell} q_{\boldsymbol{i}}(\boldsymbol{a}) (x, f_{i_1}^{(a_1)} \cdots f_{i_\ell}^{(a_\ell)})_L t_1^{a_1} \cdots t_\ell^{a_\ell},$$

where $q_{\boldsymbol{i}}(\boldsymbol{a}) := \prod_{k=1}^{\ell} q_{i_k}^{a_k(a_k-1)/2}$. Note that the all but finitely many summands in the right-hand side are zero. The map $\Phi_{\boldsymbol{i}}$ is called a Feigin homomorphism.

Feigin homomorphisms (2)

Proposition (Berenstein)

(1) *For* $\boldsymbol{i} \in I^{\ell}$, *the map* $\Phi_{\boldsymbol{i}}$ *is a* $\mathbb{Q}(q)$*-algebra homomorphism.*

(2) *For* $w \in W$ *and* $\boldsymbol{i} \in I(w)$, *we have* $\operatorname{Ker} \Phi_{\boldsymbol{i}} = \left(\mathbf{U}_{w,q}^{-}\right)^{\perp}$.

(3) *For* $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ *and* $\lambda \in P_+$, *we have*

$$\Phi_{\boldsymbol{i}}\left(D_{w\lambda,\lambda}\right) = q_{\boldsymbol{i}}(\boldsymbol{d}) t_1^{d_1} \cdots t_\ell^{d_\ell}$$

where $\boldsymbol{d} = (d_1, \dots, d_\ell)$ *with* $d_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_\ell} \lambda \rangle$.

Hence $\Phi_{\boldsymbol{i}}$ gives rise to an injective algebra homomorphism

$$\Phi_{\boldsymbol{i}} \colon \mathbf{A}_q[N_-^w] \to L_{\boldsymbol{i}}.$$

The quantum Chamber Ansatz

Theorem (O)

Let $w \in W$, $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$ *and* $k \in \{1, \ldots, \ell\}$*. Then*

$$(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})(\underline{D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}}) = \left(\prod_{j=1}^{k} q_{i_j}^{d_j(d_j+1)/2}\right) t_1^{-d_1} t_2^{-d_2} \cdots t_k^{-d_k},$$

where $d_j := \langle h_{i_j}, s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k} \rangle$ $(j = 1, \ldots, k)$*. Denote this element by* $D'^{\,(\boldsymbol{i})}_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}} \in L_{\boldsymbol{i}}$*.*

Corollary

Let $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$*. Then, for* $k \in \{1, \ldots, \ell\}$*,*

$$t_k \simeq (D'^{\,(\boldsymbol{i})}_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}})^{-1} (D'^{\,(\boldsymbol{i})}_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}})^{-1} \prod_{j \in I \setminus \{i_k\}} (D'^{\,(\boldsymbol{i})}_{w_{\leq k}\varpi_j, \varpi_j})^{-a_{j,i_k}},$$

here the right-hand side is determined up to powers of q*.*

Relation with GLS theory

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver without oriented cycles. From now on, we assume that $\mathfrak{g}$ is a symmetric Kac-Moody Lie algebra associated with Q ($Q_0 = I$). Denote by Λ the preprojective algebra corresponding to Q, that is,

$$\Lambda := \mathbb{C}\overline{Q}/(\sum_{a \in Q_1}(a^*a - aa^*)),$$

here $\mathbb{C}\overline{Q}$ is the path algebra of the double quiver of Q. For a nilpotent Λ-module X, we can define $\varphi_X \in \mathbb{C}[N_-^w]$ satisfying the following:

$$\varphi_X(y_{\boldsymbol{i}}(t_1, \dots, t_\ell)) = \sum_{\boldsymbol{a}=(a_1,\dots,a_\ell)\in\mathbb{Z}_{\geq 0}^\ell} \chi(\mathcal{F}_{\boldsymbol{i},\boldsymbol{a},X}) t_1^{a_1} \cdots t_\ell^{a_\ell},$$

here $\boldsymbol{i} \in I(w)$, χ denotes the Euler characteristic, and $\mathcal{F}_{\boldsymbol{i},\boldsymbol{a},X}$ is the projective variety of flags $X_\bullet = (X = X_0 \supset X_1 \supset \cdots \supset X_\ell = 0)$ of submodules of X such that $X_{k-1}/X_k \simeq S_{i_k}^{a_k}$ for $1 \leq k \leq \ell$ (Lusztig).

Relation with GLS theory (2)

Buan-Iyama-Reiten-Scott have constructed a 2-Calabi-Yau Frobenius subcategory $\mathcal{C}_w$ of Λ-modules, and Geiß-Leclerc-Schröer have proved that

$$\mathbb{C}[N_-^w] = \mathrm{span}_{\mathbb{C}}\{\varphi_X \mid X \in \mathcal{C}_w\}[\{\varphi_I \mid I \colon \mathcal{C}_w\text{-injective-projective}\}^{-1}].$$

Note that an object is projective in $\mathcal{C}_w$ ($\mathcal{C}_w$-projective) if and only if it is injective in $\mathcal{C}_w$ ($\mathcal{C}_w$-injective) since $\mathcal{C}_w$ is Frobenius.
For $X \in \mathcal{C}_w$, denote by $I(X)$ the injective hull of X in $\mathcal{C}_w$, and by $\Omega_w^{-1}(X)$ the cokernel of $X \to I(X)$.

$$0 \to X \to I(X) \to \Omega_w^{-1}(X) \to 0.$$

Relation with GLS theory (3)

Theorem (GLS)

Let $w \in W$. *For* $X \in \mathcal{C}_w$, $\eta_w^*(\varphi_X) = \varphi_{I(X)}^{-1}\varphi_{\Omega_w^{-1}(X)}$.

GLS have also constructed the algebra $\mathbf{A}_q[N^w]$ from $\mathcal{C}_w$ and constructed a q-analogue of φ_M, denoted by Y_M, for every reachable rigid module M. By using the theorem above, we obtain the following:

Theorem (Kimura-O)

Let $w \in W$. *For a reachable rigid module* M,

$$\eta_{w,q}(Y_M) \simeq Y_{I(M)}^{-1}Y_{\Omega_w^{-1}(M)}.$$

Corollary

Let M *as above. Then* $Y_M \in \mathbf{B}^{\mathrm{up}}$ *if and only if* $Y_{\Omega_w^{-1}(M)} \in \mathbf{B}^{\mathrm{up}}$.