Quantum twist automorphisms and quantum Chamber Ansatz formulae for unipotent cells

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Main Theme

Establish a quantum analogue of the Chamber Ansatz
– Relate Feigin homomorphisms to quantum cluster structures
– Explicit description of quantum twist automorphisms

Background ($q = 1$)

A *unipotent cell* N_-^w is the intersection $N_- \cap B_+ w B_+$ of the uniptent radical N_- of an (opposite) Borel subgroup B_- of a connected simply connected simple algebraic group G over $\mathbb{C}$ and a Schubert cell $B_+ w B_+$ in the flag variety G/B_+. Consider the following torus embedding;

$$y_{\boldsymbol{i}} \colon (\mathbb{C}^\times)^\ell \to N_-^w, (t_1, \dots, t_\ell) \mapsto \exp(t_1 F_{i_1}) \cdots \exp(t_\ell F_{i_\ell}).$$

Here $\boldsymbol{i} = (i_1, \dots, i_\ell)$ is a reduced word of w and F_i's are negative simple root vectors. This is a birational morphism from $\mathbb{C}^\ell$ to the Schubert variety X_w.

Factorization problem

Describe the inverse birational morphism $y_{\boldsymbol{i}}^{-1}$ explicitly.

This problem is solved by Berenstein, Fomin and Zelevinsky (1996, 1997), and the resulting substitutions are called the Chamber Ansatz. An important ingredient of their formulae is a twist automorphism $\eta_w \colon N_-^w \to N_-^w$.

We consider a *quantum analogue* of this story. In the quantum case, we do not have "*actual spaces*" but only have "*coordinate algebras*".

Standard notation for quantum groups

- I a finite index set, $\mathfrak{h}$ a finite dimensional $\mathbb{Q}$-vector space, P a $\mathbb{Z}$-lattice (weight lattice) of $\mathfrak{h}^*$, $P^* := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \subset \mathfrak{h}$, and $\{\alpha_i\}_{i\in I} \subset P$, $\{h_i\}_{i \in I} \subset P^*$ linearly independent subsets such that $A = (a_{ij})_{i,j\in I} = (\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a generalized Cartan matrix, $(-,-) : P \times P \to \mathbb{Q}$ an invariant symmetric $\mathbb{Z}$-bilinear form on P satisfying the following conditions:
$$(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0},\ \langle h_i, \lambda\rangle = 2\,(\lambda, \alpha_i)\,/\,(\alpha_i, \alpha_i) \ \text{ for } i \in I,\ \lambda \in P.$$
- W the Weyl group, $\{s_i\}_{i\in I}$ the simple reflections, $I(w)$ the set of reduced words of $w \in W$.
- $P_+ := \{\lambda \in P \mid \langle h_i, \lambda\rangle \geq 0 \text{ for all } i \in I\}$. Assume that there exist $\{\varpi_i\}_{i\in I} \subset P_+$ such that $\langle h_i, \varpi_j\rangle = \delta_{ij}$ (fundamental weights).
- $\mathbf{U}_q$ the quantized enveloping algebra over $\mathbb{Q}(q)$ associated with the data above, whose Chevalley generators are denoted by $\{e_i, f_i, q^h \mid i \in I, h \in P^*\}$, $\mathbf{U}_q^-$ the subalgebra of $\mathbf{U}_q$ generated by $\{f_i \mid i \in I\}$.
- $(-,-)_L \colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$ Lusztig's symmetric bilinear form.
- $\{G^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$ the dual canonical basis of $\mathbf{U}_q^-$, indexed by the Kashiwara crystal $\mathcal{B}(\infty)$.
- $V(\lambda)$ the integrable highest weight $\mathbf{U}_q$-module with a highest weight vector u_λ of weight λ, $\lambda \in P_+$. For $w \in W$, denote by $u_{w\lambda} \in V(\lambda)$ the extremal weight vector of weight $w\lambda$.
- $(-,-)_\lambda \colon V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ the nondegenerate symmetric bilinear form such that, for $u, v \in V(\lambda)$, $i \in I$ and $h \in P^*$,
$$(u_\lambda, u_\lambda)_\lambda = 1 \qquad (e_i.u, v)_\lambda = (u, f_i.v)_\lambda \qquad (q^h.u, v)_\lambda = (u, q^h.v)_\lambda.$$

Results

For $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$, set

$$\mathbf{U}_{q,w}^- := \sum\nolimits_{a_1, \cdots, a_\ell \in \mathbb{Z}_{\geq 0}} \mathbb{Q}\,(q)\, f_{i_1}^{a_1} \cdots f_{i_\ell}^{a_\ell}.$$

This space is spanned by a subset of the canonical basis, and the corresponding index subset is denoted by $\mathcal{B}_w(\infty)(\subset \mathcal{B}(\infty))$ [Kashiwara 1993].

Definition (Unipotent quantum matrix coefficients)**.** For $\lambda \in P_+$ and $u, v \in V(\lambda)$, define an element $D_{u,v} \in \mathbf{U}_q^-$ by the condition

$$(D_{u,v}, x)_L = (u, x.v)_\lambda \text{ for all } x \in \mathbf{U}_q^-.$$

For $w_1, w_2 \in W$, write $D_{w_1\lambda, w_2\lambda} := D_{u_{w_1\lambda}, u_{w_2\lambda}}$.

Definition (Quantum unipotent cells)**.** For $w \in W$, write $\underline{\mathcal{D}_w} := q^{\mathbb{Z}}\{\underline{D_{w\lambda,\lambda}} \mid \lambda \in P_+\}$. Set

$$\mathbf{A}_q[N_-^w] := (\mathbf{U}_q^- / \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}_{q,w}^-)_L = 0\})[\underline{\mathcal{D}_w}^{-1}]$$

This algebra is called a quantum unipotent cell. Here we express the quotient map $\mathbf{U}_q^- \to \mathbf{U}_q^- / \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}_{q,w}^-)_L = 0\}$ as $x \mapsto \underline{x}$.

Proposition (Kimura-O.)

Let $w \in W$. Then

$$\tilde{\mathbf{B}}_w^{\mathrm{up}} := \{q^{(\lambda, \mathrm{wt}\, b + \lambda - w\lambda)} \underline{D_{w\lambda,\lambda}}^{-1} \underline{G^{\mathrm{up}}(b)} \mid \lambda \in P_+, b \in \mathcal{B}_w(\infty)\}$$

forms a basis of $\mathbf{A}_q[N_-^w]$. We call $\tilde{\mathbf{B}}_w^{\mathrm{up}}$ the dual canonical bases of $\mathbf{A}_q[N_-^w]$.

Theorem (Kimura-O.)

Let $w \in W$. Then there exists an automorphism of the $\mathbb{Q}(q)$-algebra

$$\eta_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w],$$

given by

$$\underline{D_{u,u_\lambda}} \mapsto q^{-(\lambda, \mathrm{wt}\, u - \lambda)} \underline{D_{w\lambda,\lambda}}^{-1} \underline{D_{u_{w\lambda}, u}}$$

for all $\lambda \in P_+$ and weight vectors $u \in V(\lambda)$. Moreover $\eta_{w,q}$ is restricted to a permutation on the dual canonical basis $\tilde{\mathbf{B}}_w^{\mathrm{up}}$.

We call the map $\eta_{w,q}$ a quantum twist automorphism.

Definition (Feigin homomorphisms)**.** Let $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I^\ell$. The Laurent q-polynomial algebra $\mathcal{L}_{\boldsymbol{i}}$ is the unital associative $\mathbb{Q}(q)$-algebra generated by $t_1^{\pm 1}, \dots, t_\ell^{\pm 1}$ subject to the relations:

$$t_j t_k = q^{(\alpha_{i_j}, \alpha_{i_k})} t_k t_j \text{ for } 1 \leq j < k \leq \ell,$$
$$t_k t_k^{-1} = t_k^{-1} t_k = 1 \text{ for } 1 \leq k \leq \ell.$$

Then we can define a $\mathbb{Q}(q)$-linear map $\Phi_{\boldsymbol{i}} \colon \mathbf{U}_q^- \to \mathcal{L}_{\boldsymbol{i}}$ by

$$x \mapsto \sum\nolimits_{\boldsymbol{a} = (a_1, \dots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell} q_{\boldsymbol{i}}(\boldsymbol{a}) (x, f_{i_1}^{(a_1)} \cdots f_{i_\ell}^{(a_\ell)})_L t_1^{a_1} \cdots t_\ell^{a_\ell},$$

where $q_{\boldsymbol{i}}(\boldsymbol{a}) := \prod_{k=1}^\ell q_{i_k}^{a_k(a_k - 1)/2}$. Note that all but finitely many summands in the right-hand side are zero. The map $\Phi_{\boldsymbol{i}}$ is called a Feigin homomorphism.

Proposition (Berenstein 1996)**.** *Let $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$.*

(1) *The map $\Phi_{\boldsymbol{i}}$ is a $\mathbb{Q}(q)$-algebra homomorphism.*
(2) $\mathrm{Ker}\, \Phi_{\boldsymbol{i}} = \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}_{q,w}^-)_L = 0\}$.
(3) *For $\lambda \in P_+$, we have*

$$\Phi_{\boldsymbol{i}}(D_{w\lambda,\lambda}) = q_{\boldsymbol{i}}(\boldsymbol{d}) t_1^{d_1} \cdots t_\ell^{d_\ell},$$

where $\boldsymbol{d} = (d_1, \dots, d_\ell)$ with $d_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_\ell} \lambda\rangle$.

Hence $\Phi_{\boldsymbol{i}}$ gives rise to an injective algebra homomorphism

$$\Phi_{\boldsymbol{i}} \colon \mathbf{A}_q[N_-^w] \to \mathcal{L}_{\boldsymbol{i}}.$$

We can regard this map as a "*quantum torus embedding*".

Theorem (O.)

Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $k \in \{1, \dots, \ell\}$. Then

$$(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})(\underline{D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}}) = (\prod_{j=1}^{k} q_{i_j}^{d_j(d_j+1)/2}) t_1^{-d_1} t_2^{-d_2} \cdots t_k^{-d_k},$$

where $w_{\leq k} := s_{i_1} \cdots s_{i_k}$ and $d_j := \langle h_{i_j}, s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k}\rangle$ $(j = 1, \dots, k)$. Denote this element by $D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}} \in \mathcal{L}_{\boldsymbol{i}}$.

Corollary (The quantum Chamber Ansatz formulae)

Let $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. Then, for $k \in \{1, \dots, \ell\}$,

$$t_k \simeq (D'^{(\boldsymbol{i})}_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}})^{-1} (D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}})^{-1} \prod_{j \in I \setminus \{i_k\}} (D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_j, \varpi_j})^{-a_{j,i_k}},$$

here the right-hand side is determined up to powers of q.

Remark. If we specialize the statements at $q = 1$, they coincide with Berenstein, Fomin and Zelevinsky's formulae, while we do not use the specialization argument in our proof of the quantum Chamber Ansatz formulae.

The quantum Chamber Ansatz formulae say that

Calculating the image of the Feigin homomorphism =
Calculating the expansion with respect to the elements
$\{\eta_{w,q}^{-1}(\underline{D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}})\}_{k=1,\dots,\ell}$.

(If A is symmetric), the latter corresponds to quantum cluster expansion.

Example ($\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1)$)**.** For unipotent quantum minors $D_{2,1}$, $D_{23,12}$, $D_{3,1}$, we have

$$\eta_{w,q}^{-1}(D_{2,1}) = D_{23,12}^{-1} D_{13,12} \qquad \eta_{w,q}^{-1}(D_{23,12}) = q D_{23,12}^{-1} \qquad \eta_{w,q}^{-1}(D_{3,1}) = q D_{3,1}^{-1}.$$

By the way, $\{D_{2,1}, D_{23,12}, D_{3,1}\}$ is an initial quantum cluster of the quantum cluster algebra $\mathbf{A}_q[N_-^w]$, and $\{D_{13,12}, D_{23,12}, D_{3,1}\}$ is also a quantum cluster.

Now, we have $\Phi_{\boldsymbol{i}}(D_{2,1}) = t_1 + t_3$. By the quantum Chamber Ansatz,

$$\Phi_{\boldsymbol{i}}(D_{2,1}) = q(D'^{(\boldsymbol{i})}_{2,1})^{-1} + (D'^{(\boldsymbol{i})}_{2,1})^{-1} (D'^{(\boldsymbol{i})}_{3,1})^{-1} D'^{(\boldsymbol{i})}_{23,12}.$$

Therefore,

$$D_{2,1} = q D_{13,12}^{-1} D_{23,12} + D_{13,12}^{-1} D_{23,12} D_{3,1} D_{23,12}^{-1}$$
$$= q D_{13,12}^{-1} D_{23,12} + D_{13,12}^{-1} D_{3,1}.$$

This gives the quantum cluster expansion of the quantum cluster variable $D_{2,1}$ with respect to the quantum cluster $\{D_{13,12}, D_{23,12}, D_{3,1}\}$.

[Poster: http://www.ms.u-tokyo.ac.jp/~oya]