Twist automorphisms and Chamber Ansatz formulae for quantum unipotent cells

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Aims

Aims of this talk:

- Construct a quantum analogue of twist automorphisms on arbitrary quantum unipotent cells
  - Application: periodicity of certain twist automorphisms
- Establish a quantum analogue of the Chamber Ansatz formulae
  - Relate Feigin homomorphisms to quantum cluster structures

Introduction

Original story ($q = 1$): Consider a connected simply connected complex algebraic group G. We have a torus embedding;

$$\begin{array}{cccc} y_{\boldsymbol{i}}\colon & (\mathbb{C}^\times)^\ell & \to & N_-^w := N_- \cap B_+ w B_+ \\ & \cup\!\!\!\!\!\; \mid & & \cup\!\!\!\!\!\; \mid \\ & (t_1, \dots, t_\ell) & \longmapsto & \exp(t_1 F_{i_1}) \cdots \exp(t_\ell F_{i_\ell}). \end{array}$$

Here $\boldsymbol{i} = (i_1, \dots, i_\ell)$ is a reduced word of w. This gives a birational morphism from $\mathbb{C}^\ell$ to a Schubert variety X_w.

Problem

Describe the inverse birational morphism $y_{\boldsymbol{i}}^{-1}$.

NOTE the restriction $y_{\boldsymbol{i}}\mid_{(\mathbb{R}_{>0})^\ell}$ gives a bijection from $(\mathbb{R}_{>0})^\ell$ to "totally positive elements" in N_-^w [Lusztig].

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0 = s_1 s_2 s_1$, $\boldsymbol{i} = (1, 2, 1)$.

$$N_-^{w_0} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix} \middle| x_{31} \neq 0, x_{21}x_{32} - x_{31} \neq 0 \right\}.$$

Note that $x_{21}x_{32} - x_{31}$ is the minor corresponding to the row set $\{2, 3\}$ and the column set $\{1, 2\}$. (Such minor will be denoted by $\Delta_{23,12}$.)

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1,2,1)$, $N_-^{w_0} = \{\Delta_{3,1} \neq 0, \Delta_{23,12} \neq 0\}$.

$$y_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad y_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}$$

$$y_{\boldsymbol{i}}(t_1, t_2, t_3) = y_1(t_1) y_2(t_2) y_1(t_3) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{pmatrix}$$

Then, for $X = \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix}$, we have

$$t_1 = \frac{x_{21}x_{32} - x_{31}}{x_{32}} = \frac{\Delta_{23,12}}{\Delta_{3,2}} \quad t_2 = x_{32} = \Delta_{3,2} \quad t_3 = \frac{x_{31}}{x_{32}} = \frac{\Delta_{3,1}}{\Delta_{3,2}}.$$

This is an explicit description of $y_{\boldsymbol{i}}^{-1}$!

Introduction (2)

Formulation in terms of coordinate algebras: The torus embedding $y_{\boldsymbol{i}}$ induces an injective algebra homomorphism

$$y_{\boldsymbol{i}}^{*} \colon \mathbb{C}[N_{-}^{w}] \to \mathbb{C}[(\mathbb{C}^{\times})^{\ell}] \simeq \mathbb{C}[t_{1}^{\pm 1}, \dots, t_{\ell}^{\pm}].$$

The "description" of $y_{\boldsymbol{i}}^{-1}$ corresponds to the formula of the form;

$$\forall k,\ t_k = y_{\boldsymbol{i}}^{*}(R_k) \text{ for some (explicit) } R_k \in \mathrm{Frac}(\mathbb{C}[N_{-}^{w}]).$$

Berenstein, Fomin, Zelevinsky (1996, 1997) gave such formulae, and the resulting substitutions are called "the Chamber Ansatz". The key tool is a twist automorphism $\eta_w \colon N_{-}^{w} \to N_{-}^{w}$, which induces an algebra automorphism $\eta_w^{*} \colon \mathbb{C}[N_{-}^{w}] \to \mathbb{C}[N_{-}^{w}]$.

Main results (abstract)

There are known q-analogues $\mathbf{A}_q[N_-^w]$ and $\Phi_{\boldsymbol{i}}$ of $\mathbb{C}[N_-^w]$ and $y_{\boldsymbol{i}}$, respectively. The map $\Phi_{\boldsymbol{i}}$ is called a Feigin homomorphism.
(NOTE we do not have "actual spaces" but only have "coordinate algebras" in the setting of q-analogues.)

Theorem (Kimura-O.)

There is an algebra automorphim $\eta_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w]$, which preserves the dual canonical basis. The map $\eta_{w,q}$ is specialized to η_w^* as $q \to 1$.

By using this quantum analogue of twist automorphism, we obtain the following;

Theorem (O.)

The quantum analogue of the Chamber Ansatz formulae holds.

The Chamber Ansatz $(q = 1)$

Let

- $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ triangular decomposition (fixed),
- $\{E_i, F_i, H_i \mid i \in I\}$ Chevalley generators of $\mathfrak{g}$, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix (i.e. $[H_i, E_j] = a_{ij}E_j, \dots$),
- G connected simply connected complex algebraic group with $\mathrm{Lie}\, G = \mathfrak{g}$,
- N_-, H, N_+ closed subgroups of G such that $\mathrm{Lie}\, N_- = \mathfrak{n}_-$, $\mathrm{Lie}\, H = \mathfrak{h}$, $\mathrm{Lie}\, N_+ = \mathfrak{n}_+$,
- $B_- := N_- H$, $B_+ := HN_+$ Borel subgroups,
- $x_i(t) = \exp(tE_i)$, $y_i(t) = \exp(tF_i)$ 1-parameter subgroups corresponding to E_i, F_i,
- $W := N_G(H)/H$ Weyl group, e its unit, $\{s_i \mid i \in I\}$ simple reflections, $\ell(w)$ the length of $w \in W$,

The Chamber Ansatz $(q = 1)$

Let $\mathfrak{g}$, G, $N_{\pm}$, H, $B_{\pm}$, $x_i(t)$, $y_i(t)$, W standard notation.

- $I(w) := \{(i_1, \dots, i_{\ell(w)}) \in I^{\ell(w)} \mid w = s_{i_1} \cdots s_{i_{\ell(w)}}\}$ the set of reduced words of $w \in W$,
- $\overline{s}_i := x_i(-1)y_i(1)x_i(-1)$, $\overline{w} := \overline{s}_{i_1} \cdots \overline{s}_{i_\ell}$, $(i_1, \dots, i_\ell) \in I(w)$. In fact, $\overline{w}$ does not depend on the choice of $(i_1, \dots, i_\ell) \in I(w)$.
- $\{\varpi_i\}_{i \in I} \subset \mathrm{Hom}_{\mathrm{alg.grp.}}(H, \mathbb{C}^{\times})$ fundamental weights.
- $G_0 := N_- H N_+$, and $g = [g]_-[g]_0[g]_+$ $(g \in G_0)$ the corresponding decomposition.

The Chamber Ansatz $(q = 1)$

Let $\mathfrak{g}$, G, $N_\pm$, H, $B_\pm$, $x_i(t)$, $y_i(t)$, W, $I(w)$, $\overline{w}$, ϖ_i standard notation. Set $G_0 := N_- H N_+$, $g = [g]_-[g]_0[g]_+$ $(g \in G_0)$.

Definition (Generalized minors)

For $i \in I$, denote by $\Delta_{\varpi_i,\varpi_i}$ the regular function on G whose restriction to the open dense set G_0 is given by

$$\Delta_{\varpi_i,\varpi_i}(g) := \varpi_i([g]_0)$$

For $w_1, w_2 \in W$, define $\Delta_{w_1\varpi_i,w_2\varpi_i} \in \mathbb{C}[G]$ by

$$\Delta_{w_1\varpi_i,w_2\varpi_i}(g) = \Delta_{\varpi_i,\varpi_i}(\overline{w_1}^{-1} g \overline{w_2})$$

These elements are called *generalized minors*.

The Chamber Ansatz $(q = 1)$ (2)

For $w \in W$, set $N_-^w := N_- \cap B_+ \bar{w} B_+$ unipotent cell.

Proposition (Berenstein, Fomin, Zelevinsky)

There is a biregular morphism $\eta_w \colon N_-^w \to N_-^w$ given by

$$\eta_w(z) := [z^T \overline{w}]_-.$$

This is called a twist automorphism.

Recall the map

$$\begin{array}{cccc} y_{\boldsymbol{i}}^* \colon & \mathbb{C}[N_-^w] & \to & \mathbb{C}[t_1^{\pm 1}, \dots, t_\ell^{\pm 1}] \\ & \cup\!\!\!| & & \cup\!\!\!| \\ & f & \longmapsto & \langle f, y_{i_1}(t_1) \cdots y_{i_\ell}(t_\ell) \rangle. \end{array}$$

Here $\boldsymbol{i} = (i_1, \dots i_\ell) \in I(w)$.

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This is called a twist automorphism.

Theorem (Berenstein, Fomin, Zelevinsky)

Let $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. Set $w_{\leq m} := s_{i_1} \cdots s_{i_m}$ for $m = 1, \dots, \ell$. Then, for $k \in \{1, \dots, \ell\}$,

$$t_k = \frac{\prod_{j \in I \setminus \{i_k\}} (y_{\boldsymbol{i}}^* \circ (\eta_w^*)^{-1})(\Delta_{w_{\leq k}\varpi_j, \varpi_j})^{-a_{j,i_k}}}{(y_{\boldsymbol{i}}^* \circ (\eta_w^*)^{-1})(\Delta_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}} \Delta_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}})}.$$

These formulae are called the Chamber Ansatz.

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1)$, $N_-^{w_0} = \{\Delta_{3,1} \neq 0, \Delta_{23,12} \neq 0\}$,

$$y_{\boldsymbol{i}}(t_1, t_2, t_3) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{pmatrix}.$$

In this case,

$$\Delta_{s_1\varpi_1,\varpi_1} = \Delta_{2,1} \qquad \Delta_{s_1 s_2 \varpi_2,\varpi_2} = \Delta_{23,12} \qquad \Delta_{s_1 s_2 s_1 \varpi_1,\varpi_1} = \Delta_{3,1}.$$

Therefore, we have

$$t_1 = \frac{1}{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{2,1})} \quad t_2 = \frac{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{2,1})}{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{23,12})}$$

$$t_3 = \frac{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{23,12})}{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{2,1}\Delta_{3,1})}.$$

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1,2,1)$, $N_-^{w_0} = \{\Delta_{3,1} \neq 0, \Delta_{23,12} \neq 0\}$,

$$\Delta_{s_1\varpi_1,\varpi_1} = \Delta_{2,1} \quad \Delta_{s_1s_2\varpi_2,\varpi_2} = \Delta_{23,12} \quad \Delta_{s_1s_2s_1\varpi_1,\varpi_1} = \Delta_{3,1}.$$

Therefore, we have

$$t_1 = \frac{1}{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{2,1})} \quad t_2 = \frac{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{2,1})}{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{23,12})}$$

$$t_3 = \frac{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{23,12})}{(y_{\boldsymbol{i}}^* \circ (\eta_{w_0}^*)^{-1})(\Delta_{2,1}\Delta_{3,1})}.$$

In fact, $(\eta_{w_0}^*)^{-1}$ gives the following correspondence;

$$\Delta_{2,1} \mapsto \frac{\Delta_{3,2}}{\Delta_{23,12}} \qquad \Delta_{23,12} \mapsto \frac{1}{\Delta_{23,12}} \qquad \Delta_{3,1} \mapsto \frac{1}{\Delta_{3,1}}.$$

(Recall that

$$t_1 = \frac{\Delta_{23,12}}{\Delta_{3,2}} \quad t_2 = \Delta_{3,2} \quad t_3 = \frac{\Delta_{3,1}}{\Delta_{3,2}}.$$)

Setup

From now on, we consider quantum analogues of the story above.

- $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ a symmetrizable Kac-Moody Lie algebra($\supset$ finite dimensional simple Lie algebra) over $\mathbb{C}$ with (fixed) triangular decomposition,
- $\{\alpha_i\}_{i \in I}$ the simple roots of $\mathfrak{g}$, $\{h_i\}_{i \in I}$ the simple coroots of $\mathfrak{g}$,
- P a $\mathbb{Z}$-lattice (weight lattice) of $\mathfrak{h}^*$ and $P^* := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \subset \mathfrak{h}$ such that $\{\alpha_i\}_{i \in I} \subset P$ and $\{h_i\}_{i \in I} \subset P^*$,
- $P_+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$. Set $\langle h_i, \varpi_j \rangle = \delta_{ij}$.
- W the Weyl group of $\mathfrak{g}$ ($W \curvearrowright P, P^*$),
- $I(w)$ the set of reduced words of $w \in W$,
- $(-,-) : P \times P \to \mathbb{Q}$ a $\mathbb{Q}$-valued (W-invariant) symmetric $\mathbb{Z}$-bilinear form on P satisfying the following conditions:

$$(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}, \ \langle h_i, \lambda \rangle = 2\,(\alpha_i, \lambda)\,/\,(\alpha_i, \alpha_i) \ \text{for } i \in I, \ \lambda \in P.$$

Quantized enveloping algebra

Definition (Quantized enveloping algebras)

The quantized enveloping algebra $\mathbf{U}_q(:= \mathbf{U}_q(\mathfrak{g}))$ over $\mathbb{Q}(q)$ is the $\mathbb{Q}(q)$-algebra generated by

$$e_i, f_i \ (i \in I), \ q^h \ (h \in P^*),$$

with the following relations:
(i) $q^0 = 1, \ q^h q^{h'} = q^{h+h'}$,
(ii) $q^h e_i = q^{\langle h, \alpha_i \rangle} e_i q^h, \ q^h f_i = q^{-\langle h, \alpha_i \rangle} f_i q^h$,
(iii) $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ where $q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}$ and $t_i := q^{\frac{(\alpha_i, \alpha_i)}{2} h_i}$,
(iv) $\sum_{k=0}^{1-\langle h_i, \alpha_j \rangle} (-1)^k x_i^{(k)} x_j x_i^{(1-\langle h_i, \alpha_j \rangle - k)} = 0$ for $i \neq j$, $x = e, f$,
where $x_i^{(n)} := x_i^n / [n]_i!$, $[n]_i! := \prod_{k=1}^n (q_i^k - q_i^{-k})/(q_i - q_i^{-1})$.

Quantum unipotent subgroup

Let $\mathbf{U}_q^-$ be the subalgebra of $\mathbf{U}_q$ generated by $\{f_i\}_{i\in I}$ and $\mathbf{U}_{\mathbb{Q}[q^{\pm 1}]}^-$ the $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$ generated by $\{f_i^{(n)}\}_{i\in I, n\in\mathbb{Z}_{\geq 0}}$.

Definition

There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(-,-)_L\colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$ such that

$$(1,1)_L = 1, \qquad (f_i x, y)_L = \frac{1}{1-q_i^2}(x, e_i'(y))_L.$$

where $e_i'\colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ is the $\mathbb{Q}(q)$-linear map given by

$$e_i'(xy) = e_i'(x)\,y + q_i^{\langle \mathrm{wt}\, x, h_i\rangle} x e_i'(y), \quad e_i'(f_j) = \delta_{ij},$$

for homogeneous elements $x, y \in \mathbf{U}_q^-$.

Quantum unipotent subgroup

Let $\mathbf{U}_q^-$ be the subalgebra of $\mathbf{U}_q$ generated by $\{f_i\}_{i\in I}$ and $\mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]}$ the $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$ generated by $\{f_i^{(n)}\}_{i\in I, n\in\mathbb{Z}_{\geq 0}}$.
There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(-,-)_L\colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$.
Set

$$\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] := \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]})_L \subset \mathbb{Q}[q^{\pm 1}]\}.$$

Then $\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-]$ is a $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$.
Specialization:

$$\mathbf{U}_q^- \supset \mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]} \xrightarrow[\mathbb{C}\otimes_{\mathbb{Q}[q^{\pm 1}]}-]{\text{“}q\to 1\text{”}} \mathbf{U}(\mathfrak{n}_-)$$

$$\mathbf{U}_q^- \supset \mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] \xrightarrow[\mathbb{C}\otimes_{\mathbb{Q}[q^{\pm 1}]}-]{\text{“}q\to 1\text{”}} (\mathbf{U}(\mathfrak{n}^-))^*_{\mathrm{gr}} \simeq \mathbb{C}[N_-].$$

Quantum unipotent subgroup

Let $\mathbf{U}_q^-$ be the subalgebra of $\mathbf{U}_q$ generated by $\{f_i\}_{i\in I}$ and $\mathbf{U}_{\mathbb{Q}[q^{\pm 1}]}^-$ the $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$ generated by $\{f_i^{(n)}\}_{i\in I, n\in\mathbb{Z}_{\geq 0}}$.
There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(-,-)_L \colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$.
Set

$$\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-] := \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}_{\mathbb{Q}[q^{\pm 1}]}^-)_L \subset \mathbb{Q}[q^{\pm 1}]\}.$$

Then $\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-]$ is a $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$.

Notation

- $\mathbf{B}^{\mathrm{low}} = \{G^{\mathrm{low}}(b) \mid b \in \mathscr{B}(\infty)\}$ the Lusztig-Kashiwara's canonical/lower global basis, a $\mathbb{Q}[q^{\pm 1}]$-basis of $\mathbf{U}_{\mathbb{Q}[q^{\pm 1}]}^-$.
- $\mathbf{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(b) \mid b \in \mathscr{B}(\infty)\}$ the dual canonical/upper global basis with respect to $(-,-)_L$, a $\mathbb{Q}[q^{\pm 1}]$-basis of $\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-]$.

Quantum closed unipotent cell

Proposition (Kashiwara)

For $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$, set

$$\mathbf{U}_{q,w}^- := \sum_{a_1, \cdots, a_\ell} \mathbb{Q}(q) f_{i_1}^{a_1} \cdots f_{i_\ell}^{a_\ell}.$$

Then the following hold:

(1) *The subspace $\mathbf{U}_{q,w}^-$ does not depend on the choice of $\boldsymbol{i} \in I(w)$.*

(2) *Set $(\mathbf{U}_{q,w}^-)^\perp := \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}_{q,w}^-)_L = 0\}$. Then $(\mathbf{U}_{q,w}^-)^\perp$ is a two-sided ideal of $\mathbf{U}_q^-$.*

(3) *$(\mathbf{U}_{q,w}^-)^\perp \cap \mathbf{B}^{\mathrm{up}} (=: \{G^{\mathrm{up}}(b) \mid b \in \mathscr{B}_w(\infty)\})$ is a basis of $(\mathbf{U}_{q,w}^-)^\perp$.*

Set

$$(\mathbf{U}_{q,w}^-)^\perp_{\mathbb{Q}[q^{\pm 1}]} := \{x \in (\mathbf{U}_{q,w}^-)^\perp \mid (x, \mathbf{U}^-_{\mathbb{Q}[q^{\pm 1}]})_L \subset \mathbb{Q}[q^{\pm 1}]\},$$

Quantum closed unipotent cell (2)

Definition (Quantum closed unipotent cell)

For $w \in W$, set

$$\mathbf{A}_q[\overline{N_-^w}] := \mathbf{U}_q^-/(\mathbf{U}_{q,w}^-)^\perp = \mathbb{Q}(q) \otimes_{\mathbb{Q}[q^{\pm 1}]} \left(\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-]/(\mathbf{U}_{q,w}^-)^\perp_{\mathbb{Q}[q^{\pm 1}]} \right).$$

This is an algebra, called *a quantum closed unipotent cell*, by the proposition above.

In fact, we have

$$\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[\overline{N_-^w}] := \mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-]/(\mathbf{U}_{q,w}^-)^\perp_{\mathbb{Q}[q^{\pm 1}]} \xrightarrow[\mathbb{C}\otimes_{\mathbb{Q}[q^{\pm 1}]}-]{\text{``}q\to 1\text{''}} \mathbb{C}[\overline{N_-^w}].$$

The natural projection $\mathbf{U}_q^- \to \mathbf{A}_q[\overline{N_-^w}]$ will be written as $x \mapsto \underline{x}$.
Then $\{\underline{G^{\mathrm{up}}(b)} \mid b \in \mathscr{B}_w(\infty)\}$ is a basis of $\mathbf{A}_q[\overline{N_-^w}]$.

Unipotent quantum minors

For $\lambda \in P_+$, denote by $V(\lambda)$ the integrable highest weight $\mathbf{U}_q$-module generated by a highest weight vector u_λ of weight λ. For $w \in W$ and $\boldsymbol{i} \in I(w)$, set

$$u_{w\lambda} = f_{i_1}^{(\langle h_{i_1}, s_{i_2} \cdots s_{i_\ell} \lambda \rangle)} \cdots f_{i_{\ell-1}}^{(\langle h_{i_{\ell-1}}, s_{i_\ell} \lambda \rangle)} f_{i_\ell}^{(\langle h_{i_\ell}, \lambda \rangle)} . u_\lambda .$$

There exists a unique nondegenerate and symmetric bilinear form $(\ ,\ )_\lambda \colon V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ such that

$$(u_\lambda, u_\lambda)_\lambda = 1 \quad (e_i.u, v)_\lambda = (u, f_i.v)_\lambda \quad (q^h.u, v)_\lambda = (u, q^h.v)_\lambda$$

for $u, v \in V(\lambda)$, $i \in I$ and $h \in P^*$.

Definition (Unipotent quantum minors)

For $\lambda \in P_+$ and $u, v \in V(\lambda)$, define an element $D_{u,v} \in \mathbf{U}_q^-$ by

$$(D_{u,v}, x)_L = (u, x.v)_\lambda \text{ for arbitrary } x \in \mathbf{U}_q^- .$$

For $w_1, w_2 \in W$, write $D_{w_1\lambda, w_2\lambda} := D_{u_{w_1\lambda}, u_{w_2\lambda}}$.

Quantum unipotent cell

Proposition

Let $w \in W$*. Then* $\underline{\mathcal{D}_w} := q^{\mathbb{Z}}\{\underline{D_{w\lambda,\lambda}}\}_{\lambda \in P_+}$ *is an Ore set of* $\mathbf{A}_q[\overline{N_-^w}]$ *consisting of* q*-central elements.*

Definition (Quantum unipotent cells)

For $w \in W$, we can consider the algebras of fractions

$$\mathbf{A}_q[N_-^w] := \mathbf{A}_q[\overline{N_-^w}][\underline{\mathcal{D}_w^{-1}}]$$

by the proposition above. This is called *a quantum unipotent cell*.

Quantum unipotent cell (2)

Let $\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-^w]$ be a $\mathbb{Q}[q^{\pm 1}]$-subalgebra of $\mathbf{A}_q[N_-^w]$ generated by $\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[\overline{N_-^w}]$ and $\underline{\mathcal{D}_w^{-1}}$. Then

$$\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-^w] \xrightarrow[\mathbb{C}\otimes_{\mathbb{Q}[q^{\pm 1}]}-]{\text{“}q\to 1\text{”}} \mathbb{C}[N_-^w].$$

Proposition (Kimura-O)

Let $w \in W$. Then

$$\tilde{\mathbf{B}}_w^{\mathrm{up}} := \{q^{(\lambda, \mathrm{wt}\, b + \lambda - w\lambda)} \underline{D_{w\lambda,\lambda}}^{-1} \underline{G^{\mathrm{up}}(b)} \mid \lambda \in P_+, b \in \mathscr{B}_w(\infty)\}$$

forms a basis of $\mathbf{A}_q[N_-^w]$. Moreover $\tilde{\mathbf{B}}_w^{\mathrm{up}}$ is a $\mathbb{Q}[q^{\pm 1}]$-basis of $\mathbf{A}_{\mathbb{Q}[q^{\pm 1}]}[N_-^w]$. We call $\tilde{\mathbf{B}}_w^{\mathrm{up}}$ the dual canonical bases of $\mathbf{A}_q[N_-^w]$.

Quantum twist maps

Theorem (Kimura-O)

Let $w \in W$. Then there exists an automorphism of the $\mathbb{Q}(q)$-algebra

$$\eta_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w],$$

given by

$$\underline{D_{u,u_\lambda}} \mapsto q^{-(\lambda, \mathrm{wt}\, u - \lambda)} \underline{D_{w\lambda,\lambda}}^{-1} \underline{D_{u_{w\lambda},u}}$$

for all $\lambda \in P_+$ and weight vectors $u \in V(\lambda)$. In particular, $\eta_{w,q}$ is restricted to a permutation of $\tilde{\mathbf{B}}_w^{\mathrm{up}}$. Hence we can consider the specialization of $\eta_{w,q}$ at $q = 1$, and $\eta_{w,q} \mid_{q=1} = \eta_w^*$.

We call $\eta_{w,q}$ the quantum twist automorphism of $\mathbf{A}_q[N_-^w]$.

An application: periodicity

Assume that $\mathfrak{g}$ is finite dimensional, and let w_0 be the longest element of W.

Theorem (Kimura-O)

For a homogeneous element $x \in \mathbf{A}_q[N_-^{w_0}]$, we have

$$\eta_{w_0,q}^6(x) = q^{(\mathrm{wt}\, x + w_0\, \mathrm{wt}\, x, \mathrm{wt}\, x)} D_{w_0, -\mathrm{wt}\, x - w_0\, \mathrm{wt}\, x}\, x.$$

Here $D_{w_0,-\mathrm{wt}\, x - w_0\, \mathrm{wt}\, x} := D_{w_0\lambda_1,\lambda_1}^{-1} D_{w_0\lambda_2,\lambda_2}$ for $\lambda_1, \lambda_2 \in P_+$ with $-\mathrm{wt}\, x - w_0\, \mathrm{wt}\, x = -\lambda_1 + \lambda_2$.

When the action of w_0 on P is given by $\mu \mapsto -\mu$, the theorem above states that $\eta_{w_0,q}^6 = \mathrm{id}$ ("really" periodic). If $\mathfrak{g}$ is simple, then this condition is satisfied in the case that $\mathfrak{g}$ is of type B_n, C_n, D_{2n} for $n \in \mathbb{Z}_{>0}$ and E_7, E_8, F_4, G_2.

Remark

Remark

Recall that an arbitrary quantum unipotent cell $\mathbf{A}_q[N_-^w]$ is constructed via "quotient" and "localization" from $\mathbf{U}_q^-$. Hence $\mathbf{A}_q[N_-^w]$ is generated by $\{\underline{f_i} \mid i \in I\} \cup \{\underline{D_{w\rho,\rho}}^{-1}\}$ ($\rho := \sum_{i \in I} \varpi_i$). In particular,

$$\text{(the number of the generators of } \mathbf{A}_q[N_-^w]) \leq \#I + 1.$$

On the other hand,

$$\text{(the number of the generators of } \mathbb{C}[N_-^w]) \geq \dim N_-^w = \ell(w).$$

To check the periodicity of the twist automorphism, we only have to calculate the images of the generators under the iterated application of twist automorphism. Hence *the periodicity might be checked in the quantum setting more easily than in the classical setting*.

Remark (2)

Remark

Since $\eta_{w,q}$ preserves the dual canonical basis $\tilde{\mathrm{B}}_w^{\mathrm{up}}$, the periodicity of $\eta_{w,q}$ is the same as that of η_w.

Remark

When $\mathfrak{g}$ is finite dimensional and symmetric, the "6-periodicity" of η_{w_0} is also explained by the representation theory of preprojective algebras via "Geiss-Leclerc-Schröer's additive categorification". This periodicity corresponds to the periodicity of the syzygy functors. For arbitrary symmetric Kac-Moody cases, the periodicity of η_w is related to that of the shift functor of some triangulated category $\underline{\mathcal{C}_w}$ [Geiss-Leclerc-Schröer].

Feigin homomorphisms

We return to the quantum analogue of Chamber Ansatz formulae.

Definition (Feigin homomorphisms)

Let $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I^\ell$. The quantum torus $\mathcal{L}_{\boldsymbol{i}}$ is the unital associative $\mathbb{Q}(q)$-algebra generated by $t_1^{\pm 1}, \ldots, t_\ell^{\pm 1}$ with the relations:

$$t_k t_k^{-1} = t_k^{-1} t_k = 1 \text{ for } 1 \leq k \leq \ell,$$
$$t_j t_k = q^{(\alpha_{i_j}, \alpha_{i_k})} t_k t_j \text{ for } 1 \leq j < k \leq \ell.$$

We can define the $\mathbb{Q}(q)$-linear map $\Phi_{\boldsymbol{i}} \colon \mathbf{U}_q^- \to \mathcal{L}_{\boldsymbol{i}}$ by

$$x \mapsto \sum_{\boldsymbol{a}=(a_1,\ldots,a_\ell)\in\mathbb{Z}_{\geq 0}^\ell} q_{\boldsymbol{i}}(\boldsymbol{a})(x, f_{i_1}^{(a_1)} \cdots f_{i_\ell}^{(a_\ell)})_L t_1^{a_1} \cdots t_\ell^{a_\ell},$$

where $q_{\boldsymbol{i}}(\boldsymbol{a}) := \prod_{k=1}^{\ell} q_{i_k}^{a_k(a_k-1)/2}$. The map $\Phi_{\boldsymbol{i}}$ is called *a Feigin homomorphism*.

Feigin homomorphisms (2)

Proposition (Berenstein)

(1) *For* $\boldsymbol{i} \in I^\ell$*, the map* $\Phi_{\boldsymbol{i}}$ *is a* $\mathbb{Q}(q)$*-algebra homomorphism.*

(2) *For* $w \in W$ *and* $\boldsymbol{i} \in I(w)$*, we have* $\mathrm{Ker}\,\Phi_{\boldsymbol{i}} = \left(\mathbf{U}^-_{w,q}\right)^\perp$*.*

(3) *For* $w \in W$*,* $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$ *and* $\lambda \in P_+$*, we have*

$$\Phi_{\boldsymbol{i}}\left(D_{w\lambda,\lambda}\right) = q_{\boldsymbol{i}}(\boldsymbol{d}) t_1^{d_1} \cdots t_\ell^{d_\ell}$$

where $\boldsymbol{d} = (d_1, \ldots, d_\ell)$ *with* $d_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_\ell} \lambda \rangle$*.*

Hence $\Phi_{\boldsymbol{i}}$ gives rise to an injective algebra homomorphism

$$\Phi_{\boldsymbol{i}} \colon \mathbf{A}_q[N_-^w] \to \mathcal{L}_{\boldsymbol{i}}.$$

This is a quantum analogue of

$$y_{\boldsymbol{i}}^* \colon \mathbb{C}[N_-^w] \to \mathbb{C}[t_1^{\pm 1}, \ldots, t_\ell^{\pm 1}], f \mapsto \langle f, y_{i_1}(t_1) \cdots y_{i_\ell}(t_\ell) \rangle.$$

The quantum Chamber Ansatz

Set

$$\Phi_{\boldsymbol{i}}^{\mathrm{tw}} := \Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1} \colon \mathbf{A}_q[N_-^w] \to \mathcal{L}_{\boldsymbol{i}}.$$

We call this map *a twisted Feigin homomorphism*.
Note that $\eta_{w,q}^{-1}$ is given by $\underline{D_{u_{w\lambda},u}} \mapsto q^{(\lambda, \mathrm{wt}\, u - w\lambda)} \underline{D_{w\lambda,\lambda}}^{-1} \underline{D_{u,u_\lambda}}$.
However the description of $\eta_{w,q}^{-1}(\underline{D_{u,u_\lambda}})$ is difficult in general.

Theorem (O.)

Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ *and* $k \in \{1, \dots, \ell\}$. *Then*

$$\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(\underline{D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}}) = \left(\prod_{j=1}^{k} q_{i_j}^{d_j(d_j+1)/2} \right) t_1^{-d_1} t_2^{-d_2} \cdots t_k^{-d_k},$$

where $d_j := \langle h_{i_j}, s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k} \rangle$ $(j = 1, \dots, k)$.

The quantum Chamber Ansatz (2)

Corollary (The quantum Chamber Ansatz)

Let $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$. *Then, for* $k \in \{1, \ldots, \ell\}$,

$$t_k \simeq \frac{\prod_{j \in I \setminus \{i_k\}} \Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{w_{\leq k}\varpi_j, \varpi_j})^{-a_{j,i_k}}}{\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}} D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}})},$$

here the right-hand side is determined up to powers of q, *and* $\simeq$ *stands for the coincidence up to some powers of* q.

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1)$. Write $D_{s_1\pi_1,\pi_1} = D_{2,1}$ etc.. In type A, the unipotent quantum minors associated with the fundamental representations correspond to the q-analogues of usual minors. In this case,

$$D_{s_1\varpi_1,\varpi_1} = D_{2,1} \quad D_{s_1s_2\varpi_2,\varpi_2} = D_{23,12} \quad D_{s_1s_2s_1\varpi_1,\varpi_1} = D_{3,1}.$$

Therefore,

$$\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{2,1}) = qt_1^{-1} \quad \Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{23,12}) = q^2t_1^{-1}t_2^{-1} \quad \Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{3,1}) = q^2t_2^{-1}t_3^{-1}.$$

Hence,

$$t_1 = q\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{2,1})^{-1} \quad t_2 = q\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{23,12})^{-1}\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{2,1})$$
$$t_3 = \Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{2,1})^{-1}\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{3,1})^{-1}\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{23,12}).$$

Relation to quantum cluster algebras

Geiss-Leclerc-Schröer and Goodearl-Yakimov introduced *an quantum cluster algebra structure* on the quantum unipotent cell $\mathbf{A}_q[N_-^w]$. In these quantum cluster algebra structures, we can choose $\{D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}\}_{k=1,\dots,\ell}$ as a initial seed (up to normalization of powers of q). By *the quantum Laurent phenomenon*, we have

$$\mathbf{A}_q[N_-^w] \subset \mathbb{Q}(q)[D^{\pm 1}_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}]_{k=1,\dots,\ell} =: \mathcal{T}_{\boldsymbol{i}}.$$

Hence we have the two kinds of "quantum torus embeddings";

$$\mathcal{L}_{\boldsymbol{i}} \xleftarrow{\text{twisted Feigin homomorphism}} \mathbf{A}_q[N_-^w] \xrightarrow{\text{quantum Laurent phenomenon}} \mathcal{T}_{\boldsymbol{i}}$$

The quantum Chamber Ansatz formulae provide the explicit relation between these two embeddings!

Relation to quantum cluster algebras (2)

In other words,

Calculating the image of the twisted Feigin homomorphism "="
Calculating the cluster expansion with respect to $\{D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}\}_k$

via the quantum Chamber Ansatz formulae.
In this sense, the (non-twisted) Feigin homomorphism is also related to the cluster expansion via the quantum Chamber Ansatz formulae by the following theorem.

Theorem (Kimura-O)

Assume that $\mathfrak{g}$ *is symmetric. Let* $w \in W$. *Then* $\eta_{w,q}^{\pm 1}$ *preserve the quantum clusters (up to frozen variables).*

Calculating the image of the Feigin homomorphism "="
Calculating the cluster expansion with respect to
$\{\eta_{w,q}^{-1}(D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}})\}_k$

Relation to quantum cluster algebras (2)

Theorem (Kimura-O)

Assume that $\mathfrak{g}$ is symmetric. Let $w \in W$. Then $\eta_{w,q}^{\pm 1}$ preserve the quantum clusters (up to frozen variables).

Calculating the image of the Feigin homomorphism "$=$"
Calculating the cluster expansion with respect to
$$\{\eta_{w,q}^{-1}(D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}})\}_k$$

Remark

Our proof of this theorem strongly depends on Geiss-Leclerc-Schröer's additive categorification of (non-quantum) twist automorphisms and quantum cluster algebra structures. Hence the assumption that $\mathfrak{g}$ is symmetric is required. Conjecturally, the statement is valid also in the non-symmetric case.

Example

$\mathfrak{g} = \mathfrak{sl}_3$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1)$,

$$D_{s_1\varpi_1,\varpi_1} = D_{2,1} \quad D_{s_1 s_2\varpi_2,\varpi_2} = D_{23,12} \quad D_{s_1 s_2 s_1\varpi_1,\varpi_1} = D_{3,1}.$$

By the quantum Chamber Ansatz formulae, we have

$$\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{3,2}) = t_3^{-1}t_2^{-1}(t_1 + t_3)$$
$$= q^{-1}\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{3,1})(q\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{2,1})^{-1} + \Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{2,1})^{-1}\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{3,1})^{-1}\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{23,12})).$$

Therefore,

$$\begin{aligned} D_{3,2} &= D_{3,1}D_{2,1}^{-1} + q^{-1}D_{3,1}D_{2,1}^{-1}D_{3,1}^{-1}D_{23,12} \\ &= qD_{2,1}^{-1}D_{3,1} + D_{2,1}^{-1}D_{23,12}. \end{aligned}$$

Example (2)

Let $\mathfrak{g} = \mathfrak{sl}_4$, $w = w_0$, $\boldsymbol{i} = (1, 2, 1, 3, 2, 1)$. Then

$$\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{2,1}) = qt_1^{-1} \qquad \Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{23,12}) = q^2 t_1^{-1} t_2^{-1}$$

$$\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{3,1}) = q^2 t_2^{-1} t_3^{-1} \qquad \Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{234,123}) = q^3 t_1^{-1} t_2^{-1} t_4^{-1}$$

$$\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{34,12}) = q^4 t_2^{-1} t_3^{-1} t_4^{-1} t_5^{-1} \qquad \Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{4,1}) = q^3 t_4^{-1} t_5^{-1} t_6^{-1}$$

Now,

$$\Phi_{\boldsymbol{i}}^{\mathrm{tw}}(D_{4,3}) = t_6^{-1} t_5^{-1} t_4^{-1} (t_2 t_3 + t_2 t_6 + t_5 t_6).$$

Therefore,

$$D_{4,3} = q D_{3,1}^{-1} D_{4,1} + D_{2,1} D_{23,12}^{-1} D_{3,1}^{-1} D_{34,12} + D_{23,12}^{-1} D_{234,123}.$$