Quantum twist maps and dual canonical bases

Hironori Oya

The University of Tokyo

joint work with Yoshiyuki Kimura

Various Issues relating to Representation Theory and Non-commutative Harmonic Analysis
RIMS, 29/6/2016

Subjects

Main objects:

- Quantized enveloping algebras $\mathbf{U}_q(\mathfrak{g})$(= "$q$-analogue" of universal enveloping algebras $\mathbf{U}(\mathfrak{g})$) associated with Kac-Moody Lie algebras $\mathfrak{g}/\mathbb{C}$.
  $\xrightarrow{\text{In particular}}$
  Quantized nilpotent subalgebras $\mathbf{U}_q^-(w)$, which have special bases called dual canonical bases.
- Quantum twist maps Θ_w, introduced by Lenagan-Yakimov, which are anti-algebra isomorphisms from $\mathbf{U}_q^-(w^{-1}) \to \mathbf{U}_q^-(w)$.

(Naive) Aim: Study the compatibility between Quantum twist maps and dual canonical bases.

Main results (abstract style)

The following are brief summaries of our main results.

Theorem (Kimura-O)

The quantum twist maps induce bijections between the elements of the dual canonical bases of quantum nilpotent subalgebras.

Theorem (Kimura-O)

The quantum twist maps map some "unipotent quantum minors" to some "unipotent quantum minors".

What are "twist maps"?

History: Study of total positivity in semisimple algebraic group.
A matrix $g \in G := SL_n(\mathbb{C})$ is called totally positive (resp. nonnegative) if all minors of g are positive (resp. nonnegative).
Study of totally positive matrices has a long history (from the first half of 20th century) e.g. [Fekete 1912]

Total positivity criteria:
How many minors do we have to check in order to judge total positivity of $g \in G$?

$$\begin{array}{lcl} \text{By definition} & \rightsquigarrow & {}_{2n}C_n - 2 \text{ minors.} \\ \text{In fact} & \rightsquigarrow & n^2 - 1 \text{ minors!} \end{array}$$

More generally, we briefly explain total nonnegativity criteria following Fomin-Zelevinsky (JAMS, 1999).

What are “twist maps”? (2)

Let

- $E_{i,j}$:=the matrix unit with 1 in the (i,j)-entry, $E := \sum_{i=1}^{n} E_{i,i}$,
- $B := \{$(not strict) upper triangular matrices$\}$, $B_- := B^T$,
- $N := \{$strict upper triangular matrices$\}$, $N_- := N^T$,
- $H := \{$diagonal matrices$\} = B \cap B^-$,
- $W := N_G(H)/H \simeq \mathfrak{S}_n$ the Weyl group of $G = SL_n(\mathbb{C})$,
- For $i \in \{1, \dots, n-1\}$, set

$$x_i(t) := E + tE_{i,i+1},\ x_{\bar{i}}(t) := E + tE_{i+1,i},$$
$$\bar{s}_i := \sum_{j \neq i,i+1} E_{j,j} - E_{i,i+1} + E_{i+1,i},$$
$$\bar{\bar{s}}_i := \sum_{j \neq i,i+1} E_{j,j} + E_{i,i+1} - E_{i+1,i},$$

- For $w \in W$ and a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, set

$$\bar{w} := \bar{s}_{i_1} \cdots \bar{s}_{i_\ell},\ \bar{\bar{w}} := \bar{\bar{s}}_{i_1} \cdots \bar{\bar{s}}_{i_\ell},$$

- $G_0 := N_- H N$, $g = [g]_-[g]_0[g]_+$ ($g \in G_0$) the corresponding decomposition.

What are "twist maps"? (2)

Let $E_{i,j}$, E, B, B_-, N, N_-, H, W, $\{x_i(t), x_{\bar{i}}(t)\}_{i\in\{1,\dots,n-1\}}$, $\{\bar{w}, \bar{\bar{w}}\}_{w\in W}$, G_0, $g = [g]_-[g]_0[g]_+$ $(g \in G_0)$ standard notation.
For $u, v \in W$, set $G^{u,v} := B\bar{u}B \cap B_-\bar{v}B_-$ the double Bruhat cell.
Let $G_{\geq 0}$ the set of totally nonnegative elements, $G^{u,v}_{>0} := G_{\geq 0} \cap G^{u,v}$.
Let $H_{>0}$ the subset of H with nonnegative entries.
Let $u, v \in W$ and $u = s_{i_1} \cdots s_{i_\ell}$, $v = s_{i'_1} \cdots s_{i'_{\ell'}}$ reduced expressions.
Set $\tilde{\boldsymbol{i}} = (\tilde{i}_1, \dots, \tilde{i}_{\ell+\ell'})$:=any shuffle of $(i_1, \dots, i_\ell)$ and $(\bar{i'}_1, \dots, \bar{i'}_{\ell'})$.

Fact (Fomin-Zelevinsky)

$H \times \mathbb{C}^{\ell+\ell'} \to G^{u,v}$, $(a; t_1, \dots, t_{\ell+\ell'}) \mapsto a x_{\tilde{i}_1}(t_1) \cdots x_{\tilde{i}_{\ell+\ell'}}(t_{\ell+\ell'})$ *is a birational isomorphism* (*denoted by* $x_{\tilde{\boldsymbol{i}}}$).

Fact (Lusztig, Fomin-Zelevinsky)

The map $x_{\tilde{\boldsymbol{i}}}$ *restricts to a* *bijection* $H_{>0} \times \mathbb{R}^{\ell+\ell'}_{>0} \to G^{u,v}_{>0}$.

What are "twist maps"? (3)

Factorization problem: Find the explicit formulae for the inverse birational isomorphism $x_{\tilde{\boldsymbol{i}}}^{-1}$.
To solve this problem, Fomin-Zelevinsky introduced twist maps $\zeta^{u,v}$!

Fact (Twist maps [Fomin-Zelevinsky])

We can define a biregular isomorphism $\zeta^{u,v}\colon G^{u,v} \to G^{u^{-1},v^{-1}}$ *as*

$$\zeta^{u,v}(x) := \left([\overline{\overline{u^{-1}}}x]_{-}^{-1}\overline{\overline{u^{-1}}}x\overline{v^{-1}}[x\overline{v^{-1}}]_{+}^{-1}\right)^{\vee},$$

where $(-)^{\vee}$ *is an involution of the group* G *given by*

$$a^{\vee} = a^{-1}\ (a \in H),\ x_i(t)^{\vee} = x_{\bar{i}}(t),\ x_{\bar{i}}(t)^{\vee} = x_i(t).$$

Moreover $(\zeta^{u,v})^{-1} = \zeta^{u^{-1},v^{-1}}$.

What are "twist maps"? (3)

Factorization problem: Find the explicit formulae for the inverse birational isomorphism $x_{\tilde{\boldsymbol{i}}}^{-1}$.
To solve this problem, Fomin-Zelevinsky introduced twist maps $\zeta^{u,v}$!

Fact (Fomin-Zelevinsky)

Let $g \in G^{u,v} \cap \operatorname{Im} x_{\tilde{\boldsymbol{i}}}$. Then t_k's and the diagonal entries of a are described as the (explicit) Laurent monomials of certain minors of $\zeta^{u,v}(g)$.

By this solution, they deduce the total positivity criteria of the form:
An element $g \in G^{u,v}$ is an element of $G^{u,v}_{>0}$ if and only if certain explicit $\ell + \ell' + n - 1$ minors of g are positive.

Remark

When $u = v =$ the longest element w_0, $g \in G^{w_0,w_0}_{>0}$ is equivalent to total positivity of g.

What are "twist maps"? (3)

Factorization problem: Find the explicit formulae for the inverse birational isomorphism $x_{\tilde{\boldsymbol{i}}}^{-1}$.
To solve this problem, Fomin-Zelevinsky introduced twist maps $\zeta^{u,v}$!

Fact (Fomin-Zelevinsky)

Let $g \in G^{u,v} \cap \mathrm{Im}\, x_{\tilde{\boldsymbol{i}}}$. Then t_k's and the diagonal entries of a are described as the (explicit) Laurent monomials of certain minors of $\zeta^{u,v}(g)$.

By this solution, they deduce the total positivity criteria of the form:
An element $g \in G^{u,v}$ is an element of $G^{u,v}_{>0}$ if and only if certain explicit $\ell + \ell' + n - 1$ minors of g are positive.

Important Property

The twist map $\zeta_{u,v}$ restricts to $G^{u,v}_{>0} \to G^{u^{-1},v^{-1}}_{>0}$.

Our targets

Set $N(w) := N \cap \bar{w} N_- \bar{w}^{-1}$ and $N_-(w) := N_- \cap \bar{w} N \bar{w}^{-1}$ $(w \in W)$.

Twist maps in terms of y-coordinates

We have a biregular isomorphism

$$\begin{array}{ccc} G^{u,v} \cap G_0 & \to & (N(u) \cap G_0 \bar{u}^{-1}) \times H \times (N_-(v^{-1}) \cap \bar{v}^{-1} G_0) \\ \cup\!\!\!\!\!\!\;\;\; & & \cup\!\!\!\!\!\!\;\;\; \\ g & \longmapsto & (\bar{u}[\bar{u}^{-1} g]_- \bar{u}^{-1}, [g]_0, \bar{v}^{-1}[g \bar{v}^{-1}]_+ \bar{v}). \end{array}$$

The elements of right-hand side are denoted by (y_+, y_0, y_-) (y-coordinate). By the way, $\zeta^{u,v}$ maps $G^{u,v} \cap G_0$ to $G^{u^{-1},v^{-1}} \cap G_0$. Hence the restriction of twist maps are described using y-coordinates. Write $(y_+, y_0, y_-) \mapsto (y'_+, y'_0, y'_-)$.

Fact (Fomin-Zelevinsky)

We have $y'_- = \bar{v}(y_-^{\vee})^{-1} \bar{v}^{-1} =: \tau_v(y_-)$.

In fact $\tau_v \colon N_-(v^{-1}) \to N_-(v)$ and $\tau_v^* \colon \mathbb{C}[N_-(v)] \to \mathbb{C}[N_-(v^{-1})]$.

Our targets

Twist maps in terms of y-coordinates

We have a biregular isomorphism

$$G^{u,v} \cap G_0 \;\to\; (N(u) \cap G_0 \bar{u}^{-1}) \times H \times (N_-(v^{-1}) \cap \bar{v}^{-1} G_0)$$

The elements of right-hand side are denoted by (y_+, y_0, y_-) (y-coordinate).

Fact (Fomin-Zelevinsky)

We have $y'_- = \bar{v}(y_-^{\vee})^{-1}\bar{v}^{-1} =: \tau_v(y_-)$.

In fact $\tau_v \colon N_-(v^{-1}) \to N_-(v)$ and $\tau_v^* \colon \mathbb{C}[N_-(v)] \to \mathbb{C}[N_-(v^{-1})]$.
When proving Important Property and solving Factorization problem, they use the property "τ_u^* maps some minors to some minors".
We deal with the quantum analogue of τ_u (or τ_u^*) and prove the result corresponding to the red part generalizing minors to dual canonical bases.

Notation

Notation

Let

- $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ a symmetrizable Kac-Moody Lie algebra($\supset$ finite dimensional simple Lie algebra) over $\mathbb{C}$ with (fixed) triangular decomposition,
- $\{\alpha_i^{(\vee)}\}_{i \in I}$ the simple (co)roots of $\mathfrak{g}$,
- P a $\mathbb{Z}$-lattice (weight lattice) of $\mathfrak{h}^*$ and $P^* := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \subset \mathfrak{h}$ such that $\{\alpha_i\}_{i \in I} \subset P$ and $\{\alpha_i^\vee\}_{i \in I} \subset P^*$,
- $P_+ := \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in I\}$.
- W the Weyl group of $\mathfrak{g}$ ($W \curvearrowright P, P^*$),
- $I(w)$ the set of reduced words of $w \in W$,
- $(-,-) : P \times P \to \mathbb{Q}$ a $\mathbb{Q}$-valued (W-invariant) symmetric $\mathbb{Z}$-bilinear form on P satisfying the following conditions: $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}, \ \langle \lambda, \alpha_i^\vee \rangle = 2\,(\lambda, \alpha_i)\,/\,(\alpha_i, \alpha_i)$ for $i \in I, \ \lambda \in P$.

Quantized enveloping algebras

Definition

The quantized enveloping algebra $\mathbf{U}_q (:= \mathbf{U}_q(\mathfrak{g}))$ over $\mathbb{Q}(q)$ is the $\mathbb{Q}(q)$-algebra generated by

$$e_i, f_i \ (i \in I), \ q^h \ (h \in P^*),$$

with the following relations:
(i) $q^0 = 1, \ q^h q^{h'} = q^{h+h'}$,
(ii) $q^h e_i = q^{\langle h, \alpha_i \rangle} e_i q^h, \ q^h f_i = q^{-\langle h, \alpha_i \rangle} f_i q^h$,
(iii) $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ where $q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}$ and $t_i := q^{\frac{(\alpha_i, \alpha_i)}{2} \alpha_i^\vee}$,
(iv) $\sum_{k=0}^{1-\langle \alpha_i^\vee, \alpha_j \rangle} (-1)^k x_i^{(k)} x_j x_i^{(1-\langle \alpha_i^\vee, \alpha_j \rangle - k)} = 0$ for $i \neq j$ $(x = e, f)$,
where $x_i^{(n)} := x_i^n / [n]_i!$, $[n]_i! := \prod_{k=1}^n (q_i^k - q_i^{-k}) / (q_i - q_i^{-1})$.

Quantized enveloping algebras

Definition

The quantized enveloping algebra $\mathbf{U}_q (:= \mathbf{U}_q(\mathfrak{g}))$ over $\mathbb{Q}(q)$ is the $\mathbb{Q}(q)$-algebra generated by

$$e_i, f_i \ (i \in I), \ q^h \ (h \in P^*),$$

Relations: $q^h e_i = q^{\langle \alpha_i, h \rangle} e_i q^h$, q-Serre relations, ...
Let $\mathbf{U}_q^-$ be the subalgebra of $\mathbf{U}_q$ generated by f_i's.

Hopf algebra structure of $\mathbf{U}_q$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \ \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \ \Delta(q^h) = q^h \otimes q^h,$$
$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \varepsilon(q^h) = 1, \exists \text{antipode } S.$$

Quantized enveloping algebras

Definition

The quantized enveloping algebra $\mathbf{U}_q (:= \mathbf{U}_q(\mathfrak{g}))$ over $\mathbb{Q}(q)$ is the $\mathbb{Q}(q)$-algebra generated by

$$e_i, f_i \ (i \in I), \ q^h \ (h \in P^*),$$

Relations: $q^h e_i = q^{\langle \alpha_i, h \rangle} e_i q^h$, q-Serre relations, ...
Let $\mathbf{U}_q^-$ be the subalgebra of $\mathbf{U}_q$ generated by f_i's.

Hopf algebra structure of $\mathbf{U}_q$ (Δ, ε, S)
Let $^-\colon \mathbf{U}_q \to \mathbf{U}_q$ be the $\mathbb{Q}$-algebra involution defined by

$$\overline{q} = q^{-1}, \qquad \overline{e_i} = e_i, \qquad \overline{f_i} = f_i, \qquad \overline{q^h} = q^{-h}.$$

Let $(-)^T \colon \mathbf{U}_q \to \mathbf{U}_q$ be the $\mathbb{Q}(q)$-algebra anti-involutions defined by

$$(e_i)^T = f_i, \qquad (f_i)^T = e_i, \qquad (q^h)^T = q^h.$$

Canonical bases

Review the theory of canonical bases due to Lusztig and Kashiwara: Denote by $\left(\mathbf{U}_q^-\right)_{\mathbb{Z}}$ the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $\mathbf{U}_q^-$ generated by the elements $\{f_i^{(n)}\}_{i\in I, n\in\mathbb{Z}_{\geq 0}}$. Then there exists a free $\mathbb{Z}[q]$-submodule $\mathscr{L}(\infty)$ of $\mathbf{U}_q^-$ such that

$$\begin{array}{ccc} \left(\mathbf{U}_q^-\right)_{\mathbb{Z}} \cap \mathscr{L}(\infty) \cap \overline{\mathscr{L}(\infty)} & \xrightarrow{\text{projection}} & \mathscr{L}(\infty)/q\mathscr{L}(\infty) \\ \cup & & \cup \\ \mathbf{B}^{\mathrm{low}} & \longmapsto & \mathscr{B}(\infty) \end{array}$$

is the isomorphism of $\mathbb{Z}$-modules. Moreover we can construct a "special" $\mathbb{Z}$-basis $\mathscr{B}(\infty)$ of $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$. The inverse image of $\mathscr{B}(\infty)$ under this map is called the canonical bases $\mathbf{B}^{\mathrm{low}}$ of $\mathbf{U}_q^-$. In fact, $\mathbf{B}^{\mathrm{low}} = \{G^{\mathrm{low}}(b)\}_{b\in\mathscr{B}(\infty)}$ is a $\mathbb{Z}[q^{\pm 1}]$-basis of $\left(\mathbf{U}_q^-\right)_{\mathbb{Z}}$.

Dual canonical bases

Definition

There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_L \colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$ such that

$$(1,1)_L = 1, \qquad (f_i x, y)_L = \frac{1}{1-q_i^2}(x, e_i'(y))_L.$$

where $e_i' \colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ is the $\mathbb{Q}(q)$-linear map given by

$$e_i'(xy) = e_i'(x)\,y + q_i^{\langle \mathrm{wt}\, x, \alpha_i^\vee \rangle} x e_i'(y)\,, \quad e_i'(f_j) = \delta_{ij},$$

for homogeneous elements $x, y \in \mathbf{U}_q^-$.

Dual canonical bases

Definition

There exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_L\colon \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$.
Denote by $\mathbf{B}^{\mathrm{up}}$ the basis of $\mathbf{U}_q^-$ dual to $\mathbf{B}^{\mathrm{low}}$ with respect to the bilinear form $(\ ,\ )_L$, that is, $\mathbf{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(b)\}_{b\in\mathscr{B}(\infty)}$ such that

$$(G^{\mathrm{low}}(b), G^{\mathrm{up}}(b'))_L = \delta_{b,b'}$$

for any $b, b' \in \mathscr{B}(\infty)$.

Definition (The dual bar-involution)

For a homogeneous $x \in \mathbf{U}_q^-$, we define $\sigma(x) = \sigma_L(x) \in \mathbf{U}_q^-$ by

$$(\sigma(x), y)_L = \overline{(x, \overline{y})_L} \text{ for an arbitrary } y \in \mathbf{U}_q^-.$$

Properties

We have $\overline{G^{\mathrm{low}}(b)} = G^{\mathrm{low}}(b)$ and $\sigma(G^{\mathrm{up}}(b)) = G^{\mathrm{up}}(b)$.
Let $\left(\mathbf{U}_q^-\right)^{\mathbb{Z}} := \sum_{b \in \mathscr{B}(\infty)} \mathbb{Z}[q^{\pm 1}] G^{\mathrm{up}}(b)$. Then $\left(\mathbf{U}_q^-\right)^{\mathbb{Z}}$ is a subalgebra of $\mathbf{U}_q^-$.
Specialization:

$$\mathbf{U}_q^- \supset \left(\mathbf{U}_q^-\right)_{\mathbb{Z}} \xrightarrow[\mathbb{C}\otimes_{\mathbb{Z}[q^{\pm 1}]} -]{\text{“}q\to 1\text{”}} \mathbf{U}(\mathfrak{n}^-)$$

$$\mathbf{U}_q^- \supset \left(\mathbf{U}_q^-\right)^{\mathbb{Z}} \xrightarrow[\mathbb{C}\otimes_{\mathbb{Z}[q^{\pm 1}]} -]{\text{“}q\to 1\text{”}} \mathbb{C}[N].$$

Quantum nilpotent subalgebras

Let $T_i \circlearrowright \mathbf{U}_q$ be a certain algebra automorphism, which is a q-analogue of $\bar{s}_i \circlearrowright \mathbf{U}(\mathfrak{g})$ ($T_i = T'_{i,-1}$ in Lusztig's notation).
For $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$, we have a well-defined automorphism $T_w := T_{i_1} \cdots T_{i_\ell}$. Set

$$F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) := \prod_{k=1}^{\ell} q_{i_k}^{\frac{1}{2}c_k(c_k-1)}(1-q_{i_k}^2)^{c_k} T_{i_1} \cdots T_{i_{\ell-1}}(f_{i_\ell}^{c_\ell}) \cdots T_{i_1}(f_{i_2}^{c_2}) f_{i_l}^{c_l},$$

where $\boldsymbol{c} = (c_1, \dots, c_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}$, called an (upper) PBW-basis element.
(e.g. $\mathfrak{g} = \mathfrak{sl}_3$, $T_1(f_2) = f_1 f_2 - q f_2 f_1$.)
The set $\{F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}}$ is a linearly independent set of $\mathbf{U}_q^-$.

Fact (The quantum nilpotent subalgebras $\mathbf{U}_q^-(w)$)

Denote by $\mathbf{U}_q^-(w)$ the subspace of $\mathbf{U}_q^-$ spanned by $\{F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c}}$. In fact these subspaces are subalgebras and do not depend on the choice of $\boldsymbol{i} \in I(w)$.

Quantum nilpotent subalgebras (2)

Quantum nilpotent subalgebras are known to be compatible with dual canonical bases.

Proposition (Kimura)

Let $w \in W$ *and* $\boldsymbol{i} \in I(w)$. *Then*
(1) $\mathbf{U}_q^-(w) \cap \mathbf{B}^{\mathrm{up}}$ *is a basis of* $\mathbf{U}_q^-(w)$.
(2) *For every element* $G^{\mathrm{up}}(b)$ *of* $\mathbf{U}_q^-(w) \cap \mathbf{B}^{\mathrm{up}}$ *there uniquely exists* $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}$ *such that*
$G^{\mathrm{up}}(b) = F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) + \sum_{\boldsymbol{c}' < \boldsymbol{c}} d_{\boldsymbol{c},\boldsymbol{c}'}^{\boldsymbol{i}} F^{\mathrm{up}}(\boldsymbol{c}', \boldsymbol{i})$ *with* $d_{\boldsymbol{c},\boldsymbol{c}'}^{\boldsymbol{i}} \in q\mathbb{Z}[q]$.
Here $<$ *denotes the left lexicographic order on* $\mathbb{Z}_{\geq 0}^{\ell}$.
We denote by $b(\boldsymbol{c}, \boldsymbol{i})$ *the corresponding crystal basis element.*

In fact

$$\left(\mathbf{U}_q^-(w)\right)^{\mathbb{Z}} := \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}} \mathbb{Z}[q^{\pm 1}] G^{\mathrm{up}}(b(\boldsymbol{c}, \boldsymbol{i})) \xrightarrow[\mathbb{C} \otimes_{\mathbb{Z}[q^{\pm 1}]} -]{\text{“}q \to 1\text{”}} \mathbb{C}[N_-(w)].$$

Unipotent quantum minors

For $\lambda \in P_+$, denote by $V(\lambda)$ the integrable highest weight $\mathbf{U}_q$-module generated by a highest weight vector v_λ of weight λ. For $w \in W$ and $\boldsymbol{i} \in I(w)$, set

$$v_{w\lambda} = f_{i_1}^{(\langle s_{i_2} \cdots s_{i_\ell} \lambda, \alpha_{i_1}^\vee \rangle)} \cdots f_{i_{\ell-1}}^{(\langle s_{i_\ell} \lambda, \alpha_{i_{\ell-1}}^\vee \rangle)} f_{i_\ell}^{(\langle \lambda, \alpha_{i_\ell}^\vee \rangle)} . v_\lambda .$$

There exists a unique nondegenerate and symmetric bilinear form $(\ , \ )_\lambda \colon V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ such that

$$(v_\lambda, v_\lambda)_\lambda = 1 \qquad (x.v_1, v_2)_\lambda = (v_1, x^T . v_2)_\lambda$$

for $v_1, v_2 \in V(\lambda)$ and $x \in \mathbf{U}_q$.

Definition (Unipotent quantum minors)

For $\lambda \in P_+$ and $u, w \in W$ with $u\lambda - w\lambda \in -\sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$, define the element $D_{u\lambda, w\lambda} \in \mathbf{U}_q^-$ by the following property:

$$(D_{u\lambda, w\lambda}, x)_L = (v_{u\lambda}, x.v_{w\lambda})_\lambda \text{ for } x \in \mathbf{U}_q^- .$$

Unipotent quantum minors (2)

Proposition (Kashiwara)

The unipotent quantum minors are elements of $\mathbf{B}^{\mathrm{up}}$.

In fact $F^{\mathrm{up}}((0,\ldots,\overset{\overset{k}{\vee}}{1},\ldots,0),\boldsymbol{i})$ is a unipotent quantum minor.

Proposition

Let $\lambda \in P_+$ *and* $u_1, u_2, w \in W$. *Suppose that* u_1 *is less than or equal to* w *with respect to the weak right Bruhat order, that is, there exists* $\boldsymbol{i} = (i_1, i_2, \ldots, i_\ell) \in I(w)$ *such that* $(i_1, i_2, \ldots, i_k) \in I(u_1)$ *for some* $k \in \{1, \ldots, \ell\}$. *Then*

$$D_{u_1\lambda, u_2\lambda} \in \mathbf{U}_q^-(w).$$

Quantum twist maps

Recall the $q = 1$ case: $\tau_w(y_-) := \bar{w}(y_-^\vee)^{-1}\bar{w}^{-1}$.
Let $\vee\colon \mathbf{U}_q \to \mathbf{U}_q$ be the $\mathbb{Q}(q)$-algebra involution defined by

$$e_i^\vee = f_i, \qquad f_i^\vee = e_i, \qquad \left(q^h\right)^\vee = q^{-h}.$$

Definition (Quantum twist maps [Lenagan-Yakimov])

For $w \in W$, we consider the $\mathbb{Q}(q)$-algebra anti-automorphism Θ_w of $\mathbf{U}_q$ defined by

$$\Theta_w := T_w \circ S \circ \vee.$$

We have $(\Theta_w)^{-1} = \Theta_{w^{-1}}$. For a homogeneous element $x \in \mathbf{U}_q$, we have $\mathrm{wt}\,(\Theta_w(x)) = -w\,\mathrm{wt}\,(x)$.

Main results

Notation

(1) For $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$, we set $\boldsymbol{i}^{\mathrm{rev}} = (i_\ell, \cdots, i_1) \in I(w^{-1})$.
(2) For $\boldsymbol{c} = (c_1, \cdots, c_\ell) \in \mathbb{Z}^\ell_{\geq 0}$, we set $\boldsymbol{c}^{\mathrm{rev}} := (c_\ell, \cdots, c_1) \in \mathbb{Z}^\ell_{\geq 0}$.

Theorem (Kimura-O)

Let $w \in W$ and $\boldsymbol{i} \in I(w)$. For $G^{\mathrm{up}}(b(\boldsymbol{c}, \boldsymbol{i})) \in \mathbf{B}^{\mathrm{up}} \cap \mathbf{U}_q^-(w)$, we have

$$\Theta_{w^{-1}}\left(G^{\mathrm{up}}\left(b\left(\boldsymbol{c}, \boldsymbol{i}\right)\right)\right) = G^{\mathrm{up}}\left(b\left(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right)\right) \in \mathbf{B}^{\mathrm{up}} \cap \mathbf{U}_q^-(w^{-1}).$$

Theorem (Kimura-O)

Let $\lambda \in P_+$ and $u_1, u_2 \in W$. Suppose that u_1 and u_2 are less than or equal to w with respect to the weak right Bruhat order. Then we have

$$\Theta_{w^{-1}}(D_{u_1\lambda, u_2\lambda}) = D_{w^{-1}u_2\lambda, w^{-1}u_1\lambda}.$$

Examples

Example

For $w \in W$ and $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$, we have

$$\Theta_{w^{-1}}\left((1 - q_{i_k}^2)T_{i_1} \cdots T_{i_{k-1}}(f_{i_k})\right) = (1 - q_{i_k}^2)T_{i_\ell} \cdots T_{i_{k+1}}(f_{i_k}).$$

(We use this fact in order to prove our main theorem.)

Example

Let $\mathfrak{g} = \mathfrak{sl}_3$ and $w = s_2 s_1$. Define $\rho \in P$ by $\langle \rho, \alpha_i^\vee \rangle = 1$ $(i = 1, 2)$.
Then $D_{s_2 s_1 \rho, \rho}, D_{s_2 s_1 \rho, s_1 \rho} \in \mathbf{U}_q^-(s_2 s_1)$ by Proposition.
Now $D_{s_2 s_1 \rho, \rho}$ satisfies the assumption of our second main theorem while $D_{s_2 s_1 \rho, s_1 \rho}$ does not.
In fact $\Theta_{s_1 s_2}(D_{s_2 s_1 \rho, \rho}) = D_{s_1 s_2 \rho, \rho}$ but $\Theta_{s_1 s_2}(D_{s_2 s_1 \rho, s_1 \rho}) \neq D_{-\rho, \rho}$.
Indeed $D_{-\rho, \rho} \notin \mathbf{U}_q^-(s_1 s_2)$.

Sketch of the proof

Use the characterization of the dual canonical basis element $G^{\mathrm{up}}(b(\boldsymbol{c}, \boldsymbol{i}))$, that is, dual-bar invariance and "unitriangular property".
$\rightsquigarrow$ We check the compatibility between Θ_w and PBW-bases, the dual-bar involution.
For proving our second theorem, we use our first theorem and the following lemma (Form preserving property):

Lemma

Let $w \in W$. Then we have

$$(x, y)_L = (\Theta_{w^{-1}}(x), \Theta_{w^{-1}}(y))_L$$

for all $x, y \in \mathbf{U}_q^-(w)$.

Corollary

The following coincidence of coefficients is the immediate corollary of our results.

Corollary

Let $w \in W$ *and* $\boldsymbol{i} \in I(w)$. *Write*

$$F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) = \sum\nolimits_{\boldsymbol{c}' \in \mathbb{Z}_{\geq 0}^{\ell}} \left[F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) : G^{\mathrm{up}}(b(\boldsymbol{c}', \boldsymbol{i}))\right] G^{\mathrm{up}}(b(\boldsymbol{c}', \boldsymbol{i})).$$

Then we have

$$\left[F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) : G^{\mathrm{up}}(b(\boldsymbol{c}', \boldsymbol{i}))\right] = \left[F^{\mathrm{up}}(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}) : G^{\mathrm{up}}\left(b\left((\boldsymbol{c}')^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right)\right)\right].$$

In particular, we can write the expansion as follows:

$$F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})$$
$$= G^{\mathrm{up}}(b(\boldsymbol{c}, \boldsymbol{i})) + \sum_{\boldsymbol{c}' < \boldsymbol{c},\ (\boldsymbol{c}')^{\mathrm{rev}} < \boldsymbol{c}^{\mathrm{rev}}} \left[F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) : G^{\mathrm{up}}(b(\boldsymbol{c}', \boldsymbol{i}))\right] G^{\mathrm{up}}(b(\boldsymbol{c}', \boldsymbol{i})).$$