Similarities in the finite-dimensional representation theory for quantum affine algebras of several different types

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Representation theory

Representation theory =
Study of vectors spaces endowed with "a fixed symmetry"

"a fixed symmetry" =
various mathematical objects having "an algebraic structure"

For example,

- Groups (finite groups, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, GL_n, SL_n, SO_n, Sp_n, ...)
- Associative algebras (path algebras, coordinate algebras, quantum groups, ...)
- Lie algebras ($\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{gl}_n[t^{\pm 1}]$, $\mathfrak{sl}_n[t^{\pm 1}]$, Virasoro algebras, ...)
- Vertex operator algebras
- ...

$\rightsquigarrow$ The world of Representation theory is quite rich and extensive !

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

We review the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

A $\mathbb{C}$-vector space $\mathfrak{g}$ equipped with $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra
$\overset{\text{def}}{\Leftrightarrow}$

- $[\ ,\ ]$ is a bilinear map,
- $[x, y] = -[y, x], \forall x, y \in \mathfrak{g}$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \forall x, y, z \in \mathfrak{g}$ (Jacobi identity).

Typical example : Let V be a vector space.

$$\mathrm{End}_{\mathbb{C}}(V) \text{ is a Lie algebra by the operation } [x, y] := xy - yx$$

(This is denoted by $\mathfrak{gl}(V)$)

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

We review the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

$$\mathfrak{sl}_2(\mathbb{C}) := \{x \in \mathrm{Mat}_2(\mathbb{C}) | \mathrm{Trace}(x) = 0\}$$

$\rightsquigarrow$ $\mathfrak{sl}_2(\mathbb{C})$ is a Lie algebra by the operation $[x, y] := xy - yx$.
Explicit description Standard basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Formulas of brackets :

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h.$$

Appearance of $\mathfrak{sl}_2(\mathbb{C})$

- Cross product of vectors in the three dimensional space (over $\mathbb{C}$)

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3 \qquad \boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1 \qquad \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2.$$

$$\text{If we set } \begin{cases} e := i\boldsymbol{e}_1 + \boldsymbol{e}_2, \\ f := i\boldsymbol{e}_1 - \boldsymbol{e}_2, \\ h := -2i\boldsymbol{e}_3, \end{cases} \text{ then } \begin{cases} h \times e = 2e, \\ h \times f = -2f, \\ e \times f = h. \end{cases}$$

- Angular momentum operators (quantum mechanics)

$$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y.$$

$$\text{If we set } \begin{cases} e := (L_x + iL_y)/\hbar, \\ f := (L_x - iL_y)/\hbar, \\ h := 2L_z/\hbar, \end{cases} \text{ then } \begin{cases} [h, e] = 2e, \\ [h, f] = -2f, \\ [e, f] = h. \end{cases}$$

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$ (2)

What is a vector space V endowed with the symmetry of $\mathfrak{sl}_2(\mathbb{C})$?

$\rightsquigarrow$ V with a $\mathbb{C}$-linear map $\pi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V)$ such that

$$\pi([x,y]) = \pi(x)\pi(y) - \pi(y)\pi(x), \forall x, y \in \mathfrak{sl}_2(\mathbb{C}).$$

(Namely, a Lie algebra homomorphism $\pi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$.)

Such π is called a representation of $\mathfrak{sl}_2(\mathbb{C})$.

Easy examples of π :

- zero map $\mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V), x \mapsto 0, \forall x.$ (trivial).
- inclusion map $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathrm{Mat}_2(\mathbb{C}) \simeq \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$ (fundamental)

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$ (3)

Systematic construction : Consider the polynomial algebra $\mathbb{C}[u, v]$. As linear operators on $\mathbb{C}[u, v]$, set

$$D_e := u\frac{\partial}{\partial v} \qquad D_f := v\frac{\partial}{\partial u} \qquad D_h := u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

Then,

$$[D_e, D_f] = D_h \qquad [D_h, D_e] = 2D_e \qquad [D_h, D_f] = -2D_f.$$

Moreover, D_e, D_f, D_h preserve the degree of polynomials. Therefore, $\forall n \in \mathbb{Z}_{\geq 0}$ $\exists$ a representation $\pi_n \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[u, v]_n)$ given by

$$e \mapsto D_e \qquad f \mapsto D_f \qquad h \mapsto D_h,$$

here $\mathbb{C}[u, v]_n$ is the $(n + 1)$-dimensional subspace of $\mathbb{C}[u, v]$ spanned by the polynomials of degree n.

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$ (4)

Structure of π_n :

$$0 \overset{D_e}{\longleftarrow} u^n \underset{D_f}{\overset{D_e}{\leftrightarrows}} u^{n-1}v \underset{D_f}{\overset{D_e}{\leftrightarrows}} \cdots \underset{D_f}{\overset{D_e}{\leftrightarrows}} uv^{n-1} \underset{D_f}{\overset{D_e}{\leftrightarrows}} v^n \underset{D_f}{\longrightarrow} 0$$

Moreover $D_h.u^{n-k}v^k = (n-2k)u^{n-k}v^k$

- $u^{n-k}v^k$ is called a weight vector of weight $n-2k$ (eigenvalue of $\pi_n(h)$).
- u^n is called a highest weight vector of highest weight n.
- v^n is called a lowest weight vector of lowest weight $-n$.

Record the weights of π_n

$$\rightsquigarrow \mathrm{ch}(\pi_n) = e^{n\varpi} + e^{(n-2)\varpi} + \cdots + e^{(-n+2)\varpi} + e^{-n\varpi} \quad (e^{\varpi} : \text{symbol})$$

$$= \frac{e^{(n+1)\varpi} - e^{-(n+1)\varpi}}{e^{\varpi} - e^{-\varpi}} \quad \text{the character of } \pi_n$$

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$ (5)

Results in the representation theory of $\mathfrak{sl}_2(\mathbb{C})$

- (Semisimplicity) If $\pi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V)$ is a finite dimensional representation, then

$$V \overset{\text{always}}{=\!=} \bigoplus \underbrace{(\pi(\mathfrak{sl}_2(\mathbb{C}))\text{-stable minimal subspace})}_{\text{irreducible representation}}.$$

- (Classification) For $n \in \mathbb{Z}_{\geq 0}$, π_n is irreducible, and

$$\begin{array}{ccc} \mathbb{Z}_{\geq 0} & \overset{1:1}{\leftrightarrow} & \{\text{irreducible representation of } \mathfrak{sl}_2(\mathbb{C})\} / \simeq \\ \Cup & & \Cup \\ n & \leftrightarrow & [\pi_n] \end{array}$$

$\rightsquigarrow$ The subspace $\{v \in V \mid \pi(e).v = 0\}$ and the action of $\pi(h)$ on this space determine the whole V (dimension, basis,...) !!
In particular, $\mathrm{ch}(\pi)$ determines the isomorphism class of π.

Complex simple Lie algebras

Cartan-Killing classification of simple Lie algebras over $\mathbb{C}$ (1890's) :

- Type A_n

$$\mathfrak{sl}_{n+1}(\mathbb{C}) := \{x \in \mathrm{Mat}_{n+1}(\mathbb{C}) | \mathrm{Trace}(x) = 0\}$$

- Type B_n

$$\mathfrak{so}_{2n+1}(\mathbb{C}) := \left\{x \in \mathrm{Mat}_{2n+1}(\mathbb{C}) \middle| x^T + x = 0\right\}$$

- Type C_n

$$\mathfrak{sp}_{2n}(\mathbb{C}) := \left\{x \in \mathrm{Mat}_{2n}(\mathbb{C}) \middle| x^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} x = 0\right\}$$

- Type D_n

$$\mathfrak{so}_{2n}(\mathbb{C}) := \left\{x \in \mathrm{Mat}_{2n}(\mathbb{C}) \middle| x^T + x = 0\right\}$$

- Type $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$ (exceptional types)

Representation theory of $\mathfrak{g}$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type X_n $(\mathrm{X} = \mathrm{A}, \mathrm{B}, \dots)$.

Results in the representation theory of $\mathfrak{g}$

- (Semisimplicity) If $\pi\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ is a finite dimensional representation, then
$$V \overset{\text{always}}{=\!=\!=} \bigoplus (\pi(\mathfrak{g})\text{-stable minimal subspace}).$$
- (Classification)
$$\begin{array}{ccc} \displaystyle\sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\varpi_i & \overset{1:1}{\leftrightarrow} & \{\text{irreducible representation of } \mathfrak{g}\} / \simeq \\ \Cup & & \Cup \\ \lambda & \leftrightarrow & [\pi_\lambda] \end{array}$$

Representation theory of $\mathfrak{g}$ (2)

For a finite dimensional representation π of $\mathfrak{g}$, we can also define the character $\mathrm{ch}(\pi)$ of π.

Theorem (The Weyl character formula 1920's)

For $\lambda \in \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\varpi_i$, we have

$$\mathrm{ch}(\pi_\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})},$$

here

- W *the Weyl group,* $\ell(w)$ *the length of* w*,*
- $\rho := \sum_{i=1}^{n} \varpi_i$ *the Weyl vector,*
- Δ_+ *the set of positive roots.*

Quantum Yang-Baxter equation

We consider the deformation of the representation of $\mathfrak{g}$. $\leftarrow$ Why ?
Original motivation : Let V be a vector space. An element $R \in \mathrm{End}_{\mathbb{C}}(V^{\otimes 2})$ is said to be a solution of the quantum Yang-Baxter equation if R satisfies

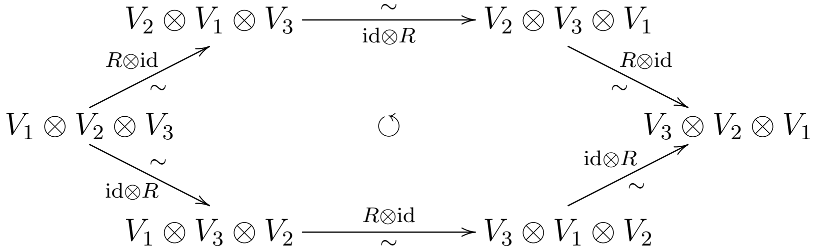
$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R)$$

in $\mathrm{End}_{\mathbb{C}}(V^{\otimes 3})$. This is a fundamental equation in the theory of integrable systems (quantum inverse scattering method).

This equation is also important in representation theory itself, knot theory, etc.

Quantum Yang-Baxter equation (2)

Basic idea for the construction of R : R is constructed as an intertwiner of tensor product representations ($V = V_1 = V_2 = V_3$):



Why representation theory is powerful ?

Philosophy : equalities among intertwiners

$\overset{\text{reduce}}{\rightsquigarrow}$ equalities among the images of highest weight vectors under intertwiners (the images of others are determined automatically by the Lie algebra symmetry !!)

Motivation of deformation

Unfortunately, representation theory of Lie algebras does not produce the *interesting* solutions...because the usual flip

$$V_1 \otimes V_2 \overset{\sim}{\to} V_2 \otimes V_1, v_1 \otimes v_2 \mapsto v_2 \otimes v_1 \tag{1}$$

already gives a nice intertwiner.
In fact, $\exists$ associative Hopf algebra $\mathcal{U}(\mathfrak{g})$ such that

$$\text{representations of } \mathfrak{g} = \text{representations of } \mathcal{U}(\mathfrak{g}).$$

This Hopf algebra $\mathcal{U}(\mathfrak{g})$ is called the universal enveloping algebra of $\mathfrak{g}$.
The coproduct of $\mathcal{U}(\mathfrak{g})$ is co-commutative. This is the reason of (1).

$$\mathcal{U}(\mathfrak{g}) \underset{\text{Hopf algebra}}{\overset{\text{not co-commutative}}{\rightsquigarrow}} \mathcal{U}_q(\mathfrak{g}) \text{ quantum group !!}$$

Quantum groups

Let $\mathfrak{g}$ be a simple Lie algebra and $q \in \mathbb{C}^\times$ not a root of 1.
$\rightsquigarrow$ We can define a quantum group $\mathcal{U}_q(\mathfrak{g})$. (Drinfeld, Jimbo mid 1980's)

The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ is the $\mathbb{C}$-algebra generated by

$$E, F, q^{\pm H},$$

with the following relations:
(i) $q^H q^{-H} = 1 = q^{-H} q^H$
(ii) $q^H E = q^2 E q^H$, $q^H F = q^{-2} F q^H$,
(iii) $[E, F] = \dfrac{q^H - q^{-H}}{q - q^{-1}}$

Coproduct Δ of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$:

$$\Delta(E) = E \otimes q^{-H} + 1 \otimes E, \ \Delta(F) = F \otimes 1 + q^H \otimes F, \ \Delta(q^H) = q^H \otimes q^H.$$

Representation theory of quantum groups

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type X_n $(\mathrm{X} = \mathrm{A}, \mathrm{B}, \dots)$.
Then the representation theory of $\mathcal{U}_q(\mathfrak{g})$ is parallel to that of $\mathfrak{g}$:

Results in the representation theory of $\mathcal{U}_q(\mathfrak{g})$

- (Semisimplicity) If $\pi^q : \mathcal{U}_q(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V)$ is a finite dimensional representation, then

$$V \overset{\text{always}}{=\!=\!=} \bigoplus (\pi(\mathcal{U}_q(\mathfrak{g}))\text{-stable minimal subspace}).$$

- (Classification)

$$\begin{array}{ccc} \displaystyle\sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\varpi_i & \overset{1:1}{\leftrightarrow} & \{\text{irreducible representation of } \mathcal{U}_q(\mathfrak{g}) \text{ of type } 1\} / \simeq \\ \Uparrow & & \Uparrow \\ \lambda & \leftrightarrow & [\pi^q_\lambda] \end{array}$$

- We can define the notion of character ch, and $\mathrm{ch}(\pi^q_\lambda)$ satisfies the Weyl character formula.

Representation theory of quantum groups (2)

Let V_1, V_2 be finite dimensional representations of $\mathcal{U}_q(\mathfrak{g})$. Then

$$V_1 \otimes V_2 \simeq V_2 \otimes V_1$$

HOWEVER, this isomorphism is not given by the usual flip but given by "universal R-matrix" !! ($\leftarrow$ this non-trivial intertwiner satisfies quantum Yang-Baxter equation.)

Example ($\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$, $V_1 = V_2 = \mathbb{C}^2$ (fundamental))

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be the canonical basis of $\mathbb{C}^2$. Take the basis of $(\mathbb{C}^2)^{\otimes 2}$ as $\{\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2, \boldsymbol{e}_2 \otimes \boldsymbol{e}_1, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2\}$. Then,

R-matrix for $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ — Usual flip

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xmapsto{q=1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Quantum loop algebras

If we consider quantum loop algebras, then we can obtain more interesting solutions :
For a simple Lie algebra $\mathfrak{g}$, we can consider the loop Lie algebra $\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ equipped with the bracket

$$[x \otimes t^m, y \otimes t^{m'}] := [x, y] \otimes t^{m+m'}.$$

The quantum loop algebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is a q-deformation of the universal enveloping algebra $\mathcal{U}(\mathcal{L}\mathfrak{g})$ of $\mathcal{L}\mathfrak{g}$. When $\mathfrak{g}$ is a simple Lie algebra of type X_n, the quantum loop algebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is said to be of type $\mathrm{X}_n^{(1)}$.

Properties

- $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ has a Hopf algebra structure.
- $\mathcal{U}_q(\mathfrak{g}) \hookrightarrow \mathcal{U}_q(\mathcal{L}\mathfrak{g})$ as a Hopf algebra.

Quantum loop algebras (2)

We again consider the case of $\mathfrak{sl}_2(\mathbb{C})$.
$\rightsquigarrow$ Any representation V of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ gives rise to

a 1-parameter family V_z $(z \in \mathbb{C}^\times)$

of representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{sl}_2(\mathbb{C}))$. *Generically*, $V_{z_1} \otimes V_{z_2} \simeq V_{z_2} \otimes V_{z_1}$
$\rightsquigarrow$ more non-trivial solutions !!

Example

In the setting of previous example, $\mathbb{C}^2_{z_1} \otimes \mathbb{C}^2_{z_2} \simeq \mathbb{C}^2_{z_2} \otimes \mathbb{C}^2_{z_1}$ gives the following R-matrix $(\xi := z_1/z_2)$:

R-matrix for $\mathcal{U}_q(\mathcal{L}\mathfrak{sl}_2(\mathbb{C}))$ $\qquad$ R-matrix for $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$

$$R(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\xi(1-q^2)}{1-\xi q^2} & \frac{q(1-\xi)}{1-\xi q^2} & 0 \\ 0 & \frac{q(1-\xi)}{1-\xi q^2} & \frac{(1-q^2)}{1-\xi q^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xmapsto{\xi=0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Quantum loop algebras (2)

Example

In the setting of previous example, $\mathbb{C}^2_{z_1} \otimes \mathbb{C}^2_{z_2} \simeq \mathbb{C}^2_{z_2} \otimes \mathbb{C}^2_{z_1}$ gives the following R-matrix ($\xi := z_1/z_2$) :

$$\underbrace{R(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\xi(1-q^2)}{1-\xi q^2} & \frac{q(1-\xi)}{1-\xi q^2} & 0 \\ 0 & \frac{q(1-\xi)}{1-\xi q^2} & \frac{(1-q^2)}{1-\xi q^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{R\text{-matrix for } \mathcal{U}_q(\mathcal{L}\mathfrak{sl}_2(\mathbb{C}))} \xmapsto{\xi=0} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{R\text{-matrix for } \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))}$$

This $R(\xi)$ gives the solution of the quantum Yang-Baxter equation with spectral parameter :

$$(R(\xi)\otimes\mathrm{id})(\mathrm{id}\otimes R(\xi\eta))(R(\eta)\otimes\mathrm{id}) = (\mathrm{id}\otimes R(\eta))(R(\xi\eta)\otimes\mathrm{id})(\mathrm{id}\otimes R(\xi))$$

(This solution is important for "the 6-vertex model".)

Quantum loop algebras (2)

This observation gives motivation to study finite dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$. However, they are rather difficult and quite different from those of $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{L}\mathfrak{g}$...

- There exists a highest weight classification of irreducible representations, BUT semisimplicity does not hold.
- There exists a notion of character (called q-character), BUT $\nexists$ known closed formulae for the q-characters of irreducible representations in general.
- Sometimes, $V \otimes W \not\simeq W \otimes V$. Note that $R(\xi)$ has a pole at $\xi = q^{-2}$. Indeed, if $z_1/z_2 = q^{-2}$, then $\mathbb{C}^2_{z_1} \otimes \mathbb{C}^2_{z_2} \not\simeq \mathbb{C}^2_{z_2} \otimes \mathbb{C}^2_{z_1}$.

Research theme

Let $\mathcal{C}_{\mathrm{X}_n^{(1)}}$ be the category of finite dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ of type $\mathrm{X}_n^{(1)}$.

Recently, similarities among $\mathcal{C}_{\mathrm{X}_n^{(1)}}$ have been recognized.
(Frenkel-Hernandez, Kashiwara-Kim-Oh, Oh-Scrimshaw, Hernandez-O.,. . .)

Observation :

$$\mathcal{C}_{\mathrm{A}_{2n-1}^{(1)}} \overset{\text{similar}}{\sim} \mathcal{C}_{\mathrm{B}_n^{(1)}} \qquad \mathcal{C}_{\mathrm{D}_{n+1}^{(1)}} \overset{\text{similar}}{\sim} \mathcal{C}_{\mathrm{C}_n^{(1)}}$$

$$\mathcal{C}_{\mathrm{E}_6^{(1)}} \overset{\text{similar}}{\sim} \mathcal{C}_{\mathrm{F}_4^{(1)}} \qquad \mathcal{C}_{\mathrm{D}_4^{(1)}} \overset{\text{similar}}{\sim} \mathcal{C}_{\mathrm{G}_2^{(1)}}$$

Hope : Become able to study the category $\mathcal{C}_{\mathrm{X}_m^{(1)}}$ by using its paired category $\mathcal{C}_{\mathrm{Y}_n^{(1)}}$!!

There are no known direct relations between the quantum loop algebras of type $\mathrm{X}_m^{(1)}$ and $\mathrm{Y}_n^{(1)}$ themselves.

Main result

Notation

For an (artinian and noetherian) monoidal abelian category $\mathcal{A}$, write its Grothendieck ring as $K(\mathcal{A})$.

- spanned by the isomorphism classes of $\mathrm{Obj}(\mathcal{A})$,
- short exact sequences in $\mathcal{A} \rightsquigarrow$ addition in $K(\mathcal{A})$,
- tensor product in $\mathcal{A} \rightsquigarrow$ product in $K(\mathcal{A})$.

Theorem (Kashiwara-Oh '17, Oh-Scrimshaw '18)

For each pair $\mathcal{C}_{\mathrm{X}_m^{(1)}} \sim \mathcal{C}_{\mathrm{Y}_n^{(1)}}$ appearing above, $\exists$ "not-small" monoidal abelian subcategories $\mathcal{C}_{\mathcal{Q},\mathrm{X}_m^{(1)}} \subset \mathcal{C}_{\mathrm{X}_m^{(1)}}$, $\mathcal{C}_{\mathcal{Q}',\mathrm{Y}_n^{(1)}} \subset \mathcal{C}_{\mathrm{Y}_n^{(1)}}$ such that

$$K(\mathcal{C}_{\mathcal{Q},\mathrm{X}_m^{(1)}}) \overset{\text{algebra}}{\simeq} K(\mathcal{C}_{\mathcal{Q}',\mathrm{Y}_n^{(1)}}), \left\{ \begin{array}{c} q\text{-characters} \\ \text{of irred. rep's} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} q\text{-characters} \\ \text{of irred. rep's} \end{array} \right\}.$$

Main result

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There exists a "t-deformation" $K_t(\mathcal{C}_{\mathrm{X}_m^{(1)}})$ of $K(\mathcal{C}_{\mathrm{X}_m^{(1)}})$ for all X.

Theorem (Hernandez-O. '18, arXiv:1803.06754)

$$K_t(\mathcal{C}_{\mathcal{Q},\mathrm{A}_{2n-1}^{(1)}}) \overset{\text{algebra}}{\simeq} K_t(\mathcal{C}_{\mathcal{Q}',\mathrm{B}_n^{(1)}}), \left\{ \begin{array}{c} (q,t)\text{-characters} \\ \text{of irred. rep's} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} (q,t)\text{-characters} \\ \text{of irred. rep's} \end{array} \right\}.$$

Moreover, we gave an explicit correspondence of irreducible representations in terms of highest weights.

Quantum Grothendieck rings

What is a "t-deformation" $K_t(\mathcal{C}_{X_m^{(1)}})$ of $K(\mathcal{C}_{X_m^{(1)}})$?

$\rightsquigarrow$ A $\mathbb{Z}[t^{\pm 1/2}]$-algebra with $K_{t=1}(\mathcal{C}_{X_m^{(1)}}) = K(\mathcal{C}_{X_m^{(1)}})$ introduced by

- ($\mathrm{X} = \mathrm{ADE}$ case) Nakajima '04 in a geometric way via quiver varieties
- (X : arbitrary) Hernandez '04 in an algebraic way
  $\nexists$ geometry for non-ADE cases

$\rightsquigarrow$ $K_t(\mathcal{C}_{X_m^{(1)}})$ provides algorithm to compute the "(q,t)-characters" of irreducible representations ! "Kazhdan-Lusztig algorithm"

This algorithm essentially uses the bar-involution $\overline{(\ \cdot\ )}$, $t^{1/2} \mapsto t^{-1/2}$.

Quantum Grothendieck rings

What is a "t-deformation" $K_t(\mathcal{C}_{\mathrm{X}_m^{(1)}})$ of $K(\mathcal{C}_{\mathrm{X}_m^{(1)}})$?

$\rightsquigarrow$ $K_t(\mathcal{C}_{\mathrm{X}_m^{(1)}})$ provides algorithm to compute the "(q,t)-characters" of irreducible representations ! "Kazhdan-Lusztig algorithm"

This algorithm essentially uses the bar-involution $\overline{(\ \cdot\ )}$, $t^{1/2} \mapsto t^{-1/2}$.

Theorem (Nakajima (ADE cases), Hernandez (arbitrary))

There exists a "relatively easy" $\mathbb{Z}[t^{\pm 1/2}]$-basis $\{M_t(m) \mid m \in \mathcal{B}\}$ of $K_t(\mathcal{C}_{\mathrm{X}_m^{(1)}})$.

$\rightsquigarrow$ $\exists!\{L_t(m) \mid m \in \mathcal{B}\}$ a $\mathbb{Z}[t^{\pm 1/2}]$-basis of $K_t(\mathcal{C}_{\mathrm{X}_m^{(1)}})$ such that

(1) $\overline{L_t(m)} = L_t(m)$, *and*

(2) $M_t(m) = L_t(m) + \sum_{m'<m} P_{m,m'}(t) L_t(m')$ *with* $P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$.

The element $L_t(m)$ is called the (q,t)-character of irred. rep.

(1) and (2) provide an inductive algorithm for computing $P_{m,m'}(t)$'s.

Quantum Grothendieck rings

What is a "t-deformation" $K_t(\mathcal{C}_{X_m^{(1)}})$ of $K(\mathcal{C}_{X_m^{(1)}})$?

$\rightsquigarrow$ A $\mathbb{Z}[t^{\pm 1/2}]$-algebra with $K_{t=1}(\mathcal{C}_{X_m^{(1)}}) = K(\mathcal{C}_{X_m^{(1)}})$ introduced by

- ($\mathrm{X} = \mathrm{ADE}$ case) Nakajima '04 in a geometric way via quiver varieties
- (X : arbitrary) Hernandez '04 in an algebraic way
  $\nexists$ geometry for non-ADE cases

$\rightsquigarrow$ $K_t(\mathcal{C}_{X_m^{(1)}})$ provides algorithm to compute the "(q, t)-characters" of irreducible representations ! "Kazhdan-Lusztig algorithm"
This algorithm essentially uses the bar-involution $\overline{(\ \cdot\ )}$, $t^{1/2} \mapsto t^{-1/2}$.

Nakajima's geometric construction guarantees that
the $(q, 1)$-character of irred. rep. $=$ the q-character of irred. rep.

In non-ADE cases, the $(q, 1)$-characters are still candidates of the q-characters.

Application of our main result

Theorem (Hernandez-O.)

The (q,t)-characters of irreducible representations in $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$ specialize to the corresponding q-characters at $t=1$.

Strategy of proofs : Prove that our isomorphism specialize to Kashiwara-Oh's isomorphism at $t=1$.

A priori,

- Kashiwara-Oh's isomorphism maps the q-characters of irred. rep's in $\mathcal{C}_{\mathcal{Q},\mathrm{A}_{2n-1}^{(1)}}$ to the q-characters of irred. rep's in $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$,
- Our isomorphism at $t=1$ maps the $(q,1)$-characters (= the q-characters $\leftarrow$ type A !) of irred. rep's in $\mathcal{C}_{\mathcal{Q},\mathrm{A}_{2n-1}^{(1)}}$ to the $(q,1)$-characters of irred. rep's in $\mathcal{C}_{\mathcal{Q},\mathrm{B}_n^{(1)}}$.

Reference : arXiv:1803.06754