# Representations of quantized function algebras and the transition matrices from Canonical bases to PBW bases

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# Outline


1. Preliminaries
2. Our main theorem
3. Quantized function algebras
4. Our strategy for proving positivity

# Quantized enveloping algebras

- $\mathfrak{g}$ a finite dimensional complex simple Lie algebra
- $U_q(\mathfrak{g}) = \langle E_i, F_i, K_i | i \in I \rangle_{\mathbb{Q}(q)\text{-algebra}}$
  the quantized enveloping algebra/$\mathbb{Q}(q)$ (a $q$-analogue of $U(\mathfrak{g})$)
- $U_q(\mathfrak{n}^+) = \langle E_i | i \in I \rangle_{\mathbb{Q}(q)\text{-algebra}}$

The quantized enveloping algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure. In particular, its coproduct $\Delta$ is defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \Delta(F_i) = F_i \otimes K_{-i} + 1 \otimes F_i,\ \Delta(K_i) = K_i \otimes K_i.$$

# Some operations

## Definition

We define the $\mathbb{Q}(q)$-algebra involution $\omega : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ by

$$\omega(E_i) = F_i,\ \omega(F_i) = E_i,\ \omega(K_i) = K_i^{-1}.$$

We define the $\mathbb{Q}(q)$-algebra anti-involution $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ by

$$*(E_i) = E_i,\ *(F_i) = F_i,\ *(K_i) = K_i^{-1}.$$

We define the $\mathbb{Q}$-algebra involution $\overline{(\cdot)} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ by

$$\overline{E_i} = E_i,\ \overline{F_i} = F_i,\ \overline{K_i} = K_i^{-1},\ \overline{q} = q^{-1}.$$

# PBW bases

Let $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ be a reduced word of the longest element $w_0$ of the Weyl group $W$. (i.e. $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$. In particular, $N :=$ the length of $w_0$.)

## Definition (The PBW bases)

The vectors

$$\left\{ E_{\mathbf{i}}^{\mathbf{c}} := E_{i_1}^{(c_1)} T'_{i_1,1}(E_{i_2}^{(c_2)}) \cdots T'_{i_1,1} T'_{i_2,1} \cdots T'_{i_{N-1},1}(E_{i_N}^{(c_N)}) \right\}_{\mathbf{c}}$$

$\left(\mathbf{c} = (c_1, c_2, \ldots, c_N) \in (\mathbb{Z}_{\geqq 0})^N\right)$ forms a basis of $U_q(\mathfrak{n}^+)$. Here, $T'_{i,1}$ is a $q$-analogue of "the action of the braid group".

## Remark

For any reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ of $w_0$, we have

$$\Delta_+ = \{\beta_{\mathbf{i}}^1, \beta_{\mathbf{i}}^2, \ldots, \beta_{\mathbf{i}}^N\} \text{ where } \beta_{\mathbf{i}}^k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}).$$

# Canonical bases

Let $\mathbf{i}$ be a reduced word of $w_0$. Then, there uniquely exists a basis $\{ G^{(\mathbf{c})} \}_{\mathbf{c}}$ of $U_q(\mathfrak{n}^+)$ such that

- $\overline{G^{(\mathbf{c})}} = G^{(\mathbf{c})}$,
- $G^{(\mathbf{c})} = E_{\mathbf{i}}^{\mathbf{c}} + \sum_{\mathbf{d}>\mathbf{c}} {}_{\mathbf{i}}\zeta_{\mathbf{d}}^{\mathbf{c}} E_{\mathbf{i}}^{\mathbf{d}}$ with ${}_{\mathbf{i}}\zeta_{\mathbf{d}}^{\mathbf{c}} \in q\mathbb{Z}[q]$.

We consider the lexicographic order on $(\mathbb{Z}_{\geqq 0})^N$.

## Definition (The canonical basis)

We call $\{ G^{(\mathbf{c})} \}_{\mathbf{c}}$ the canonical basis of $U_q(\mathfrak{n}^+)$.

## Remark

The definition of canonical basis does not depend on the choice of $\mathbf{i}$. (The data $(\mathbf{c})$ depend on $\mathbf{i}$.)

# Our main theorem

## Theorem (Positivity)

*Assume that the Lie algebra $\mathfrak{g}$ is of type $ADE$. Take an arbitrary reduced word $\mathbf{i}$ of $w_0$. Then, for any $\mathbf{c} \in (\mathbb{Z}_{\geqq 0})^N$, we have*

$$G^{(\mathbf{c})} = E_{\mathbf{i}}^{\mathbf{c}} + \sum_{\mathbf{d} > \mathbf{c}} {}_{\mathbf{i}}\zeta_{\mathbf{d}}^{\mathbf{c}} E_{\mathbf{i}}^{\mathbf{d}} \text{ with } {}_{\mathbf{i}}\zeta_{\mathbf{d}}^{\mathbf{c}} \in q\mathbb{N}[q].$$

## Remark

In general, it is difficult to describe the explicit form of the element of the canonical basis.

# About this theorem

The positivity of these coefficients was originally proved by

- Lusztig (1990) : for the “adapted” reduced word $\mathbf{i}$ of $w_0$ (via “geometric realization”)
- Kato (2014) : for the arbitrary case (via “categorification”)

We gave a new algebraic proof of the above theorem from now on. (It has been recently found that our “calculation procedure” is same as a certain other calculation procedure.)
From now on, we again assume that $\mathfrak{g}$ is a finite dimensional complex simple Lie algebra.

# Quantized function algebras

The dual space $U_q(\mathfrak{g})^*$ of $U_q(\mathfrak{g})$ has a $\mathbb{Q}(q)$-algebra structure induced from the coalgebra structure of $U_q(\mathfrak{g})$.

## Definition (The quantized function algebra)

The quantized function algebra $\mathbb{Q}_q[G]$ is a subalgebra of $U_q(\mathfrak{g})^*$ generated (in fact, spanned) by the matrix coefficients

$$c^{\lambda}_{f,v} \mapsto (u \mapsto \langle f, u.v \rangle),$$

here,

- $\lambda \in P_+ (=$ the set of dominant integral weight),
- $V(\lambda)$ the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda$,
- $f \in V(\lambda)^*, v \in V(\lambda)$.

Then, $\mathbb{Q}_q[G]$ has a Hopf algebra structure induced from the one of $U_q(\mathfrak{g})$ and a left and right $U_q(\mathfrak{g})$-algebra structure.

# Representation of $\mathbb{Q}_q[G]$

$\mathbb{Q}_q[G]$ - a quantum analogue of the algebra of regular functions on $G$ ($G$ is the connected simply connected simple complex algebraic group whose Lie algebra is $\mathfrak{g}$.)
The algebra $\mathbb{Q}_q[G]$ has infinite dimensional irreducible modules [This point is extremely different from the classical($=$"$q = 1$") situation!!]:

$$\mathbb{Q}_q[G] \twoheadrightarrow \mathbb{Q}_{q_i}[SL_2] \curvearrowright V_i := \bigoplus_{m \in \mathbb{Z}_{\geqq 0}} \mathbb{Q}(q)|m\rangle\rangle_i.$$

(dual to $U_{q_i}(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{g})$.)

# Representation of $\mathbb{Q}_q[G]$

**Theorem (Soibelman (1990))**

*Let $w \in W$. Then, for any reduced expression $w = s_{i_1} \cdots s_{i_l}$, the $\mathbb{Q}_q[G]$-module $V_{i_1} \otimes \cdots \otimes V_{i_l}$ is irreducible and its isomorphism class does not depend on the choice of the reduced expressions.*
*Hence, we denote this module by $V_w$.*

# Strategy 1

## Theorem (O-)

*For* $x \in U_q(\mathfrak{n}^+), \lambda \in P_+$ *and a reduced word* $\mathbf{i}$ *of* $w_0$*, we write*

$$x = \sum_{\mathbf{c} \in (\mathbb{Z}_{\geqq 0})^N} {}_{\mathbf{i}}\zeta^x_{\mathbf{c}} E^{\mathbf{c}}_{\mathbf{i}} \text{ with } {}_{\mathbf{i}}\zeta^x_{\mathbf{c}} \in \mathbb{Q}(q), \text{ and}$$

$$(c^\lambda_{f_\lambda, v_{w_0\lambda}}. * (x)).|(0)\rangle\rangle_{\mathbf{i}} = \sum_{\mathbf{c} \in (\mathbb{Z}_{\geqq 0})^N} {}_{\mathbf{i}}\zeta^{\lambda, x}_{\mathbf{c}} |(\mathbf{c})\rangle\rangle_{\mathbf{i}} \text{ with } {}_{\mathbf{i}}\zeta^{\lambda, x}_{\mathbf{c}} \in \mathbb{Q}(q) (\text{in } V_{w_0}).$$

*(Here,* $f_\lambda$ *is a highest weight vector of* $V(\lambda)^*$ *which sends a fixed highest weight vector* $v_\lambda$ *of* $V(\lambda)$ *to 1, and* $v_{w_0\lambda}$ *is the lowest weight lower global basis element of* $V(\lambda)$*.)*
*When* $\lambda \in P_+$ *tends to* $\infty$ *in the sense that* $\langle \lambda, \alpha_i^\vee \rangle$ *tends to* $\infty$ *for all* $i \in I$*,* ${}_{\mathbf{i}}\zeta^{\lambda, x}_{\mathbf{c}}$ *converges to* ${}_{\mathbf{i}}\zeta^x_{\mathbf{c}}$ *in the complete discrete valuation field* $\mathbb{Q}((q))$*.*

# Strategy 2

For sufficiently large $L$, we set $\lambda_0 := 2(N+1)L\rho$. $\rho :=$ the Weyl vector. Then, by the calculation method of the previous theorem, we can obtain

$$\left(c^{\lambda_0}_{f_{\lambda_0}, v_{w_0\lambda_0}} . * (G^{(\mathbf{c})})\right) .|(0)\rangle\rangle_{\mathbf{i}} = \sum_{\mathbf{d} \geqq \mathbf{c}} {}_{\mathbf{i}}\zeta^{\mathbf{c}}_{\mathbf{d}} |(\mathbf{d})\rangle\rangle_{\mathbf{i}} + q^L \sum_{\mathbf{d}' \in (\mathbb{Z}_{\geqq 0})^N} \eta_{\mathbf{d}'} |(\mathbf{d}')\rangle\rangle_{\mathbf{i}}$$

with $\eta_{\mathbf{d}'} \in \mathbb{Z}[q]$.

$$\left(=: \sum_{\mathbf{d} \in (\mathbb{Z}_{\geqq 0})^N} {}_{\mathbf{i}}\zeta'^{\mathbf{c}}_{\mathbf{d}} |(\mathbf{d})\rangle\rangle_{\mathbf{i}}\right)$$

# Strategy 3

On the other hand, we calculate the left-hand side of the previous equality as follows:

$$
\begin{aligned}
&(c^{\lambda_0}_{f_{\lambda_0}, v_{w_0\lambda_0}} . * (G^{(\mathbf{c})})).|(0)\rangle\rangle_{\mathbf{i}} \\
&\quad = \sum_{b'_1,\dots,b'_{N-1} \in B(\lambda_0)} c^{\lambda_0}_{(G^{\mathrm{low}}_{\lambda_0}(b'_0),\cdot), G^{\mathrm{up}}_{\lambda_0}(b'_1)} .|0\rangle\rangle_{i_1} \otimes c^{\lambda_0}_{(G^{\mathrm{low}}_{\lambda_0}(b'_1),\cdot), G^{\mathrm{up}}_{\lambda_0}(b'_2)} .|0\rangle\rangle_{i_2} \\
&\qquad\qquad \otimes \cdots \otimes c^{\lambda_0}_{(G^{\mathrm{low}}_{\lambda_0}(b'_{N-1}),\cdot), v_{w_0\lambda_0}} .|0\rangle\rangle_{i_N}.
\end{aligned}
$$

Here,

- $(\ ,\ ) : V(\lambda_0) \times V(\lambda_0) \to \mathbb{Q}(q)$ the "good" $\mathbb{Q}(q)$-bilinear form
- $\left\{ G^{\mathrm{low/up}}_{\lambda_0}(b') \right\}_{b' \in B(\lambda_0)}$ the lower/upper global basis of $V(\lambda_0)$
- $G^{\mathrm{low}}_{\lambda_0}(b'_0) := \omega(G^{(\mathbf{c})}).v_{\lambda_0}$

# Strategy 4

We can deduce that:

**Proposition**

*For each* $k$, $c^{\lambda_0}_{(G^{\mathrm{low}}_{\lambda_0}(b'_{k-1}),\cdot),G^{\mathrm{up}}_{\lambda_0}(b'_k)}.|0\rangle\rangle_{i_k} = p_k|c\rangle\rangle_{i_k}$,
*with* $c := -\frac{1}{2}\langle \mathrm{wt}\, b'_{k-1} + \mathrm{wt}\, b'_k, \alpha^\vee_{i_k}\rangle$ *and* $p_k \in q^{-L}\mathbb{Z}[q]$

- ${}_{\mathbf{i}}\zeta'^{\mathbf{c}}_{\mathbf{d}} \in \mathbb{Z}[q^{\pm 1}]$, and
- we may ignore the degree $\geqq NL$ part of the Laurent polynomial $p_k$ for any $k$ when calculating the degree $< L$ parts of the Laurent polynomials ${}_{\mathbf{i}}\zeta'^{\mathbf{c}}_{\mathbf{d}}$.

Key: "the positivity of $q$-derivations" (Lusztig)

# Remark

In our calculation, we use the following property of canonical bases:

## Proposition (Similarity of the structure constants)

*We set*

$$F_i^{(p)} G^{\mathrm{low}}(b) = \sum_{\tilde{b} \in B(\infty)} c^{\tilde{b}}_{-pi,b} G^{\mathrm{low}}(\tilde{b}),$$

$$(e_i')^p (G^{\mathrm{low}}(b)) = \sum_{\tilde{b} \in B(\infty)} \widehat{d}^{i,p}_{b,\tilde{b}} G^{\mathrm{low}}(\tilde{b}).$$

*Then, for any* $b, \hat{b} \in B(\infty), i \in I$ *and* $p \in \mathbb{Z}_{\geqq 0}$, *we have*

$$\left( c^{\hat{b}}_{-pi,b} \right)_{< -\Delta_i (d-1)p} = \left( q_i^{\frac{1}{2}d(d-1)} \begin{bmatrix} \varepsilon_i(\hat{b}) \\ p \end{bmatrix}_i \widehat{d}^{i,d}_{b, \tilde{e}_i^{\varepsilon_i(\hat{b})} \hat{b}} \right)_{< -\Delta_i (d-1)p},$$

*where* $d := \varepsilon_i(\hat{b}) - p$.

Reference: arXiv1501.01416 (Slides: http://www.ms.u-tokyo.ac.jp/˜oya)